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The Time Consistency of the Friedman Rule



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The time consistency of the Friedman Rule

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ABSTRACT

We consider the problem of time consistency of the Ramsey monetary and fiscal policies in an economy without capital. Following Lucas and Stokey (1983) we allow the government at date t to leave its successor at $t + 1$ a profile of real and nominal debt of all maturities, as a way to influence its decisions. We show that the Ramsey policies are time consistent if and only if the Friedman rule is the optimal Ramsey policy.

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A classic issue in macroeconomics is the time consistency of optimal monetary and fiscal policy. In a seminal contribution, Calvo (1978) showed that the incentive for the government to inflate away its nominal liabilities leads to a time inconsistency problem. We consider an infinite horizon model with money in the utility function of the representative consumer. The government has access to nominal and real debt of all maturities and must finance a given stream of government expenditures with a combination of consumption taxes and seignorage. We show that with a sufficiently rich maturity structure of real and nominal debt, optimal policy is time consistent if and only if the Friedman rule holds, in the sense that nominal interest rates are zero.

Our approach to the problem of time inconsistency follows that of Lucas and Stokey (1983). We begin by solving for the Ramsey policies in which the government is presumed to have a commitment technology that binds the actions of future government and hence chooses policy once and for all. These Ramsey policies consist of sequences of consumption taxes and money supplies. We then consider an environment without such a commitment technology in which each government inherits a given maturity structure of nominal and real debt. Each such government then decides on the current setting for the consumption tax and the money supply as well as the maturity structure of nominal and real debt that its successor will inherit. We ask whether it is possible to choose the maturity structure of real and nominal debt so that each government carries out the Ramsey policies. If it is we call the Ramsey policies *time consistent*.

Our model with both nominal and real bonds has two forces, nominal and real, that lead to time inconsistency. These stem from the desire to minimize the need to raise revenues with distortionary taxes. As in Calvo (1978), if the government has nominal liabilities it is

tempted to reduce the real value of these liabilities with a surprise inflation or a change in the term structure of nominal interest rates. Conversely, if the government has nominal assets it is tempted to increase the real value of these assets with a surprise deflation or a change in the term structure of nominal interest rates. As in Lucas and Stokey (1983), when the government has real liabilities or real assets it is tempted to reduce the value of its liabilities or increase the value of its assets with an appropriate change in the real term structure of interest rates. We ask, as do Lucas and Stokey (1983) and Persson, Persson and Svensson (1987), if we can nullify these forces with a careful choice of the maturity structure of both bonds and hence make the Ramsey policies time consistent.

We begin by showing that if the Friedman rule does not hold the Ramsey policies cannot be made time consistent. Briefly, when the Friedman rule does not hold the nominal debt passed on to successor governments must be restricted greatly. These restrictions curtail the ability of each government to influence the actions of its successor so much that they cannot induce them to continue with the Ramsey policies. When the Friedman rule holds, these restrictions are not binding and each government can induce its successor to continue with the Ramsey policies.

More specifically, we proceed as follows. In Lemma 1 we derive the Ramsey problem and in Proposition 1 we give sufficient conditions for the Friedman rule to solve the Ramsey problem. In this problem, the interesting initial conditions are those that lead the present value of the initial nominal liabilities to be zero. Given such initial conditions, in Lemma 2 we show that in any period t in which the nominal interest rate is positive the value of the nominal debt from date t on must be zero. If not the government could make the present value of nominal liabilities from date 0 on negative by either raising or lowering the date

t interest rate, and then turn them into arbitrarily large real assets by making the initial price level low enough. In Proposition 2 we show that when the Friedman rule does not hold, the restrictions on nominal debt derived in Lemma 2 are so severe that the Ramsey allocations cannot be made time consistent. In essence, the restrictions from Lemma 2 force each government to rely mainly on real debt to influence its successors and this debt by itself is not rich enough to control the two forces that lead to time inconsistency.

In Proposition 3 we show that if the Friedman rule holds in each period, the Ramsey problems can be made time consistent. Briefly, when the Friedman rule holds in each period agents become satiated with money balances and marginal changes in interest rates have no effect on their behavior. In this sense, the effect of money essentially disappears from the economy. The real debt can then be set so as to nullify the forces from the real side of the economy that lead to time inconsistency and the Ramsey policies can be made time consistent. We show that there are multiple ways to set the nominal debt, including setting it to zero in each period.

For most of the paper we follow the original approach to time inconsistency used by Calvo (1978), Lucas and Stokey (1983), and Persson, Persson and Svensson (1987). We then relate this definition to that of sustainable plans by Chari and Kehoe (1992) and credible policies by Stokey (1991) which explicitly builds the lack of commitment of the government into the environment with an equilibrium concept in which governments explicitly think through how their choices of debt influence the choices of their successors. In Proposition 4, we show that the Ramsey policies are time consistent if and only if they are supportable as a Markov sustainable equilibrium.

We develop a useful analogy between our monetary model and a real economy of Lucas

and Stokey (1983). That real economy has leisure and two types of consumption goods as well as two real bonds that payoff in units of the two consumption goods. Lucas and Stokey show that if the maturity structure of both bonds are carefully chosen the Ramsey policies can be made time consistent. Moreover, it is easy to show that if there is only one such bond the Ramsey policies are time consistent if and only if it a uniform consumption tax is optimal. Our monetary economy has leisure and two other goods, a consumption good and real balances as well as a bond that pays off in units of the consumption good and one that pays off in units of money. If we restrict the nominal bond to be one-period-lived there is essentially only one real bond through which governments can influence the future. In such a case, the Ramsey policies are time consistent if and only if the Friedman rule is optimal in each period. The conditions for with the Friedman rule are optimal in the monetary economy are the same as the conditions for uniform taxation to be optimal in the real economy.

Our paper is also closely related to that of Persson, Persson and Svensson (1987). They argued that with a sufficiently rich term structure of both nominal and real debt, optimal policy can be made time consistent regardless of whether or not the Friedman rule is satisfied. Unfortunately, their result is not true. Calvo and Obstfeld (1990) argued that the mistake made by Persson, Persson and Svensson (1987) was that their solution violated the unchecked second order conditions. Our analysis shows that their mistake has nothing to do with the second order conditions, rather it has to do with a lack of attention to subtle corners. Moreover, in contrast to the conjectures by Calvo and Obstfeld (1990) the Persson, Persson and Svensson (1987) conclusion can be rescued for environments in which the Friedman rule is optimal.

1. The economy and the Ramsey problem

Consider a monetary economy with money, nominal debt and real debt. Time is discrete. The economy is endowed with a linear technology described by

$$(1) \quad c_t + g_t = l_t,$$

where c_t , g_t , l_t denote consumption, government spending and labor at t . Throughout the sequence of government spending is exogenously given.

Consumers have preferences over sequences of consumption c_t , (end-of-period) real balances M_t/p_t , and labor l_t given by

$$(2) \quad \sum_{t=0}^{\infty} \beta^t U(c_t, M_t/p_t, l_t)$$

with the discount factor $0 < \beta < 1$. Here M_t is nominal balances and p_t is the nominal price level and we let $m_t = M_t/p_t$ denote real balances. We assume the period utility function $U(c, m, l)$ is concave, twice continuously differentiable, increasing in c , weakly increasing in m and increasing in l , where here and throughout the paper we denote partial derivatives by U_m , U_{mm} and so on. We also assume that consumers are satiated at a finite level of real balances, so that for each value of c and l there is a finite level of m such that $U_m(c, m, l) = 0$.

In terms of assets, we assume there are both nominal and real bonds for each maturity at each date. For the nominal bonds, for each t and s with $t \leq s$, we let $Q_{t,s}$ denote the price of one dollar at time s in units of dollars at time t and let $B_{t,s}$ denote the number of such nominal claims. Likewise for the real bonds we let $q_{t,s}$ denote the price of one unit of consumption at time s in units of consumption at time t and let $b_{t,s}$ denote the number of such real bonds. We let $B_t = (B_{t,t+1}, B_{t,t+2}, \dots)$ denote the sequence of nominal bonds purchased at t which pay off $B_{t,s}$ at time s for all $s \geq t + 1$. We use similar notation for the

real bonds b_t , the nominal and real debt prices Q_t and q_t . For later use it is worth noting that arbitrage among these bonds implies that for all $t \leq r \leq s$ their prices satisfy $Q_{t,s} = Q_{t,r}Q_{r,s}$, $q_{t,s} = q_{t,r}q_{r,s}$, $Q_{t,s} = q_{t,s}p_t/p_s$. By convention, $Q_{t,t} = 1$ and $q_{t,t} = 1$.

The consumer's sequence budget constraint at date t can be written

$$(3) \quad p_t(1+\tau_t)c_t + M_t + \sum_{s=t+1}^{\infty} Q_{t,s}B_{t,s} + p_t \sum_{s=t+1}^{\infty} q_{t,s}b_{t,s} = p_t l_t + M_{t-1} + \sum_{s=t}^{\infty} Q_{t,s}B_{t-1,s} + p_t \sum_{s=t}^{\infty} q_{t,s}b_{t-1,s}$$

Thus, at date t , the consumer has inherited nominal balances M_{t-1} , the sequence of nominal bonds B_t , and the sequence of real bonds b_t and nominal wage income of $p_t l_t$. The consumer purchases consumption c_t , new money balances M_t , and new vectors of nominal bonds B_t and real bonds b_t . Purchases of consumption are taxed at rate τ_t . At date 0 consumers have initial money balances M_{-1} , together with initial sequences of nominal and real debt claims, B_{-1} and b_{-1} . We assume that the real value of both nominal and real debt is bounded by some arbitrarily large constants.

It is convenient to work with the consumer's problem in its date 0 form. To that end, note that the sequence of budget constraints can be collapsed to the date 0 budget constraint

$$(4) \quad \sum_{t=0}^{\infty} q_{0,t} [(1+\tau_t)c_t + (1-Q_{t,t+1})m_t] = \sum_{t=0}^{\infty} q_{0,t} l_t + \frac{M_{-1}}{p_0} + \sum_{t=0}^{\infty} Q_{0,t} \frac{B_{-1,t}}{p_0} + \sum_{t=0}^{\infty} q_{0,t} b_{-1,t}.$$

The consumer's problem at date 0 is to choose sequences of consumption, real balances and leisure to maximize (2) subject to the (4).

The government's sequence budget constraint at date t is

$$(5) \quad \sum_{s=t+1}^{\infty} Q_{t,s}B_{t,s} + p_t \sum_{s=t+1}^{\infty} q_{t,s}b_{t,s} = \sum_{s=t}^{\infty} Q_{t,s}B_{t-1,s} + p_t \sum_{s=t}^{\infty} q_{t,s}b_{t-1,s} + p_t g_t - (M_t - M_{t-1}) - p_t \tau_t c_t$$

At date t , the government inherits the nominal liabilities M_{t-1} and B_t , and real liabilities b_t . To finance government spending g_t it collects consumption taxes $\tau_t c_t$ and issues new money

supply M_t , new nominal liabilities B_t , and new real liabilities b_t . Using the resource constraint and the consumer's date 0 budget constraint we can collapse these sequence of constraints into the government's date 0 budget constraint

$$(6) \quad \sum_{t=0}^{\infty} q_{0,t} [\tau_t c_t + (1 - Q_{t,t+1}) m_t] = \sum_{t=0}^{\infty} q_{0,t} g_t + \frac{M_{-1}}{p_0} + \sum_{t=0}^{\infty} Q_{0,s} \frac{B_{-1,s}}{p_0} + \sum_{t=0}^{\infty} q_{0,t} b_{-1,t}$$

We find it convenient to use the notation ${}_t c = (c_t, c_{t+1}, \dots)$ for consumption and other variables as well. We then have, for given initial conditions M_{-1}, B_{-1} , and b_{-1} , a *competitive equilibrium* is a collection of sequences of consumption, real balances and labor $({}_0 c, {}_0 m, {}_0 l)$ together with sequences of prices $({}_0 p, Q_0, q_0)$ and taxes ${}_0 \tau$ that satisfy the resource constraint and consumer maximization at date 0. The government budget constraint is then implied.

We show that the allocations in a competitive equilibrium are characterized by two simple conditions: the resource constraint and the *implementability constraint* which is given by

$$(7) \quad \sum_{t=0}^{\infty} \beta^t [c_t U_{c_t} + m_t U_{m_t} + l_t U_{l_t}] = -\frac{U_{l_0}}{p_0} [M_{-1} + \sum_{t=0}^{\infty} Q_{0,t} B_{-1,t}] - \sum_{t=0}^{\infty} \beta^t U_{l_t} b_{-1,t}$$

where $Q_{0,t} = \prod_{s=0}^{t-1} (1 + U_{m_s}/U_{l_s})$. This implementability constraint should be thought of as the date 0 budget constraint of either the consumer or the government where the consumer's first order conditions have been used to substitute out prices and policies.

Lemma 1. The consumption, real balances and labor allocations together with the price p_0 of a date 0 competitive equilibrium necessarily satisfy the resource constraint and the implementability constraint. Furthermore, given such allocations and a price p_0 that satisfy these constraints, we can construct nominal money supplies, prices and real and nominal debt prices such that these allocations and prices constitute a date 0 competitive equilibrium.

Proof. We first show that a competitive equilibrium must satisfy the implementability constraint in addition to the resource constraint. The necessary conditions for the consumer's problem include

$$(8) \quad \beta^t U_{ct} = \theta q_{0t}(1 + \tau_t)$$

$$(9) \quad \beta^t U_{mt} = \theta q_{0t}(1 - Q_{t,t+1})$$

$$(10) \quad -\beta^t U_{lt} = \theta q_{0t}$$

where θ is the multiplier on the consumer's date 0 budget constraint. Recall throughout that U_{lt} is the derivative with respect to labor and is negative. Notice that (8) and (9) imply

$$(11) \quad Q_{t,t+1} = 1 + U_{mt}/U_{lt}$$

so that

$$(12) \quad Q_{0,t} = Q_{0,1}Q_{1,2} \dots Q_{t-1,t} = (1 + U_{m0}/U_{l0})(1 + U_{m1}/U_{l1}) \dots (1 + U_{m,t-1}/U_{l,t-1})$$

Multiplying the budget constraint by θ and using (8)-(10) and (12) gives the implementability constraint.

Next, we suppose that we are given allocations and a p_0 that satisfy the resource constraint and the implementability constraint. We construct the rest of the competitive equilibrium as follows. From the first-order conditions we have

$$(13) \quad (1 + \tau_t) = -U_{ct}/U_{lt}$$

$$(14) \quad q_{0,t} = \beta^t U_{lt}/U_{l0}$$

and from the arbitrage condition for nominal and real debt prices we have

$$(15) \quad p_t = \frac{q_{0t}}{Q_{0,t}} p_0.$$

>From (12), (13), and (14) we construct $Q_{0,t}$, τ_t and $q_{0,t}$. Given these we then use (15) and p_0 to construct p_t . The nominal money supply is then given by $M_t = p_t m_t$. The initial price level p_0 is given from the implementability constraint. *Q.E.D.*

Consider now the *Ramsey problem at date 0*, given M_{-1} , B_{-1} and b_{-1} , namely to choose c_t , m_t , l_t and p_0 to solve

$$(16) \quad \max \sum_{t=0}^{\infty} \beta^t U(c_t, m_t, l_t)$$

subject to the resource constraint, the implementability constraint, and the nonnegativity constraint $p_0 \geq 0$.

Our results will depend critically on whether or not the Ramsey allocations satisfy the *Friedman rule* in that

$$(17) \quad Q_{t,t+1} = 1 \text{ for all } t,$$

so that nominal interest rates are zero in each period. Since $Q_{t,t+1} = 1 + U_{mt}/U_{lt}$ and $U_{lt} < 0$ it follows that the Friedman rule holds if and only if

$$(18) \quad U_{mt} = 0 \text{ for all } t.$$

Notice for later use that under our assumptions on preferences that if $U_m = 0$ then

$$(19) \quad U_{mc} = U_{mm} = U_{ml} = 0.$$

To see this note that U is weakly increasing in m , twice continuously differentiable, and concave and U_m is weakly decreasing in m . Thus if $U_m = 0$ at some point m it is also equal

to zero at all points $m' \geq m$ and since it is twice continuously differentiable $U_{mm} = 0$. To see that $U_{cm} = 0$ notice that by concavity $U_{cc}U_{mm} - U_{cm}^2 \geq 0$, so $U_{cm} = 0$ and a similar argument applies for U_{ml} .

We now discuss the initial conditions for both real and nominal debt that we choose for the Ramsey problem. To make the problem interesting we want initial conditions for which distortionary taxes are necessary. In particular, we want to avoid conditions under which the government can attain the *lump-sum tax allocations*. These allocations are defined to be those that solve the lump-sum tax problem of maximizing utility subject to only the resource constraint. These allocations are characterized by the following conditions for all t : no tax distortions, $-U_{lt}^*/U_{ct}^* = 1$, the Friedman rule, $U_{mt}^* = 0$, and the resource constraint, where U_{ct}^* , U_{lt}^* and U_{mt}^* denote the marginal utilities evaluated at the lump-sum tax allocations.

Clearly, if the present value of the government's initial real claims on consumers is higher than the present value of its spending then there is no need to resort to distortionary taxation. Thus, to make the optimal taxation problem interesting we will assume this is not the case. Throughout the rest of the paper we assume that *real revenues are needed* in that

$$(20) \quad \sum_{t=0}^{\infty} \beta^t U_{ct}^* (g_t + b_{-1,t}) > 0$$

holds. Equation (20) requires that at the prices consistent with the lump-sum tax allocations the present value of the sum government spending and its initial real liabilities is positive.

The solution to the Ramsey problem depends critically on the structure of the value of the initial nominal liabilities as well, namely the initial money supply M_{-1} and the vector of initial nominal debt claims B_{-1} through the term

$$(21) \quad -\frac{U_{l0}}{p_0} [M_{-1} + \sum_{t=0}^{\infty} Q_{0,t} B_{-1,t}]$$

in the implementability constraint. The term $[M_{-1} + \sum_{t=0}^{\infty} Q_{0,t} B_{-1,t}]$ is the present value of nominal liabilities of the government in units of dollars at date 0. Dividing by p_0 converts this value into date 0 consumption good units and multiplying by $-U_{l_0}$ converts it into date 0 marginal utility units.

We assume that at date 0 *nominal liabilities are zero in each period*, so that

$$(22) \quad M_{-1} + B_{-1,0} = 0 \text{ and } B_{-1,t} = 0 \text{ for all } t \geq 1$$

holds then the present value of nominal liabilities in (21) is identically zero and the Ramsey problem is independent of p_0 .

In the next section we show that the Ramsey problem is time consistent if and only if the Friedman rule is optimal in each period. Here we establish sufficient conditions for the Friedman rule to be optimal in each period for an economy that has initial nominal claims zero in each period. Under (22), the Ramsey problem at date 0 is to choose ${}_0c$, ${}_0l$, and ${}_0m$ to maximize utility (16) subject to the resource constraint and the implementability constraint

$$(23) \quad \sum_{t=0}^{\infty} \beta^t [c_t U_{ct} + m_t U_{mt} + l_t U_{lt}] = - \sum_{t=0}^{\infty} \beta^t U_{lt} b_{-1,t}.$$

Letting λ_0 and γ_t denote the multipliers on the implementability constraint and the resource constraint, the first order conditions for c_t , m_t and l_t are

$$(24) \quad U_{ct}(1 + \lambda_0) + \lambda_0 [c_t U_{cct} + m_t U_{mct} + (l_t + b_{-1,t}) U_{lct}] = \gamma_t,$$

$$(25) \quad U_{mt}(1 + \lambda_0) + \lambda_0 [c_t U_{cmt} + m_t U_{mmt} + (l_t + b_{-1,t}) U_{lmt}] = 0,$$

$$(26) \quad U_{lt}(1 + \lambda_0) + \lambda_0 [c_t U_{clt} + m_t U_{mlt} + (l_t + b_{-1,t}) U_{llt}] = -\gamma_t.$$

We can use these first order conditions to establish circumstances under which the Friedman rule is optimal. Consider an economy with preferences that are *separable and homothetic* in that

$$(27) \quad U(c, m, l) = u(w(c, m)) + V(l)$$

where w is homothetic in c and m and for which initial nominal claims are zero period by period. Then we can adapt the logic of Chari, Christiano, and Kehoe (1996) to show the following.

Proposition 1. If preferences are separable and homothetic so that (27) holds and the initial nominal claims are zero in each period so that (22) holds then the Friedman rule is optimal.

Proof. Using separability, we can arrange the first order conditions for real money balances and consumption to be

$$(28) \quad U_{mt} \left\{ (1 + \lambda_0) + \lambda_0 \left[\frac{c_t U_{cmt} + m_t U_{mmt}}{U_{mt}} \right] \right\} = 0$$

$$(29) \quad U_{ct} \left\{ (1 + \lambda) + \lambda_0 \left[\frac{c_t U_{cct} + m_t U_{cmt}}{U_{ct}} \right] \right\} = \gamma_t$$

Now in (28) either $U_{mt} = 0$ or the expression in brackets is equal to 0. We claim that $U_{mt} = 0$.

To see this note that since $U(c, m) = u(w(c, m))$ with w homothetic it follows that

$$\frac{U_m(\alpha c, \alpha m)}{U_c(\alpha c, \alpha m)} = \frac{U_m(c, m)}{U_c(c, m)}$$

for a positive constant α . Differentiating this with respect to α and evaluating it at $\alpha = 1$ gives that

$$(30) \quad \frac{c_t U_{cmt} + m_t U_{mmt}}{U_{mt}} = \frac{c_t U_{cct} + m_t U_{cmt}}{U_{ct}}$$

Hence the expression in brackets in (28) equals the expression in brackets in (29). But since the multiplier on the resource constraint γ_t is strictly positive the term in brackets in both (28) and (29) is strictly positive, so $U_{mt} = 0$. *Q.E.D.*

It is worth pointing out that this lemma is related to but not covered by the result in Chari, Christiano, and Kehoe (1996) which proved that the Friedman rule holds in an environment with weakly separable preferences, labor income taxes and no initial government debt. If we add to that environment long-lived initial real government debt then the Friedman result typically does not hold. Here we have additively separable preferences, consumption taxes and long-lived initial real government debt and the Friedman rule holds. (Briefly, with labor income taxes adding initial real debt introduces terms of the form $U_{cct}b_{-1,t}$ and $U_{cmb_{-1,t}}$ that make the proof in Chari, Christiano and Kehoe not work. With labor income taxes having real debt introduces terms of the form $-U_{lct}b_{-1,t}$ and $-U_{lmt}b_{-1,t}$ which disappear with additive separability.)

2. Time consistency and the Friedman rule

In this section we give a version of Lucas and Stokey's definition of time consistency and establish that, given certain conditions on initial nominal assets, the Ramsey problem is time consistent if and only if the Friedman rule holds.

We begin with a definition of time consistency. It is convenient to define the *Ramsey problem at date t* , given inherited values for money balances M_{t-1} , nominal debt B_{t-1} and real debt b_{t-1} to be the problem of choosing allocations from period t onward, namely $\{c_t, l_t,$

and ${}_t m$, and the price level p_t (by choosing M_t) to maximize

$$\max \sum_{s=t}^{\infty} \beta^s U(c_s, m_s, l_s)$$

subject to the resource constraint for $s \geq t$ and the implementability constraint at t

$$(31) \quad \sum_{s=t}^{\infty} \beta^{s-t} [c_s U_{c_s} + m_s U_{m_s} + l_s U_{l_s}] = -\frac{U_{l_t}}{p_t} [M_{t-1} + \sum_{s=t}^{\infty} Q_{t,s} B_{t-1,s}] - \sum_{s=t}^{\infty} \beta^{s-t} U_{l_s} b_{t-1,s}].$$

where $Q_{t,s} = \prod_{r=t}^{s-1} (1 + U_{m_r}/U_{l_r})$. The Ramsey problem at date t is said to be *time consistent* for $t+1$ if there exist values for nominal money balances M_t , nominal debt B_t , and real debt b_t with $M_t = p_t m_t$ such that taking these values as given, the continuation of the Ramsey allocations at period t from $t+1$ on, namely ${}_{t+1}c$, ${}_{t+1}l$, and ${}_{t+1}m$ together with the price level p_{t+1} solve the Ramsey problem at $t+1$ where the price level p_{t+1} is a function of the allocations and the nominal money supply according to

$$p_{t+1} = \frac{Q_{t,t+1} p_t}{q_{t,t+1}} = \beta \frac{U_{l_{t+1}}}{U_{l_t}} \frac{p_t}{(1 + U_{m_t}/U_{l_t})}.$$

The Ramsey problem at date 0 is *time consistent* if the Ramsey problem at date t is time consistent for $t+1$ for all $t \geq 0$.

Given this definition, the way to establish that a Ramsey problem at say date 0 is time consistent for date 1 is to show how the initial conditions for the date 1 problem, namely M_0 , B_0 , and b_0 , can be chosen so as to give incentives for the government at date 1 to continue with the allocations chosen by the government at date 0.

Since the solution of the Ramsey problem at 0 entails an interior solution for p_1 with $0 < p_1 < \infty$ the first order conditions of the Ramsey problem at date 1 imply that for the Ramsey plan at date 0 to be time consistent it is necessary that M_0 and B_0 satisfy the

condition

$$(32) \quad M_0 + \sum_{t=1}^{\infty} Q_{1,t} B_{0,t} = 0.$$

This condition, however, is not sufficient to eliminate the nominal forces that lead to time inconsistency. Here we present an intermediate result which shows that restrictions on the type of nominal debt sequences that can be part of the Ramsey policies when the Friedman rule does not hold. This lemma will apply to the Ramsey problem at any date $t = 0, 1, \dots$. For concreteness, consider the Ramsey problem at date 1 with inherited initial conditions M_0 , B_0 , and b_0 . If at some date s , the Friedman rule does not hold so that

$$(33) \quad Q_{s,s+1} < 1,$$

then the present value of debt from date $s + 1$ on must be zero, so that

$$(34) \quad 0 = \sum_{t=s+1}^{\infty} Q_{1,t} B_{-1,t} = Q_{1s} Q_{s,s+1} \sum_{t=s}^{\infty} Q_{s+1,t} B_{-1,t}.$$

We will assume there is some date, say r , in which strictly positive taxes are being levied so that $-U_{cr}/U_{lr} = (1 + \tau_r) > 1$. We will assume that at this date r , the second derivatives satisfies the conditions

$$(35) \quad U_{mm} + U_{lm} < 0 \text{ if } U_m > 0, \quad U_{ll} + U_{lc} \geq 0, \quad U_{mc} + U_{ml} \geq 0.$$

We show the following.

Lemma 2. If at the solution to the Ramsey problem at date 1, $0 < p_1 < \infty$, there is some date s at which the Friedman rule does not hold, there is some date r at which consumption taxes are levied and the above conditions (35) on second derivatives hold then the value of initial nominal debt from $s + 1$ on is zero so that (34) holds.

Proof. We establish the result by showing that if the assumptions of the lemma hold and (34) does not hold, then we can perturb the allocations and increase utility.

If (32) and (33) holds and (34) does not hold then we can make the present value of the government's nominal liabilities negative by a small change in $Q_{s,s+1}$. This change, which may entail either raising or lowering $Q_{s,s+1}$, is feasible since the Friedman rule does not hold at s , rather (33) holds. We do so by changing c_s, m_s and l_s in a way that satisfies the resource constraint but changes $Q_{s,s+1}$. Then by lowering the initial price level p_1 , we can generate any level of real assets for the government net of the cost of government spending.

In the second part of the perturbation we simply change the allocation at that date r in which we know positive taxes are being raised in a way that raises utility at that date and holds fixed the $Q_{r,r+1}$ so that we know the first part of the perturbation still works. To that end note that positive taxes at date r implies that $-U_{lr} < U_{cr}$. Since $U_{mr} \geq 0$ we can increase c_r and m_r and decrease l_r in a way that keeps $Q_{r,r+1}$ constant, satisfies the resource constraint, and increases utility at date r . Clearly, by the implicit function theorem for a fixed $Q_{r,r+1}$ and g_r there exist functions $l(c)$ and $m(c)$ such that $c, l(c)$ and $m(c)$ satisfy

$$(36) \quad U_m(c, m(c), l(c)) + (1 - Q_{r,r+1})U_l(c, m(c), l(c)) = 0$$

and $c + g_r = l(c)$. These functions satisfy $l'(c) = 1$ and if $U_m > 0$ then

$$(37) \quad m'(c) = - \left[\frac{U_{mc} + U_{ml} + (1 - Q_{r,r+1})(U_{ll} + U_{lc})}{U_{mm} + (1 - Q_{r,r+1})U_{lm}} \right]$$

which is nonnegative under our assumptions on second derivatives. (Note that since $U_{mm} \leq 0$ then even if $U_{cm} > 0$ the denominator in (37) is nonnegative since $U_{mm} + U_{lm} < 0$ and $1 - Q_{r,r+1} \leq 1$.)

If $U_m > 0$ then increasing c and thus $l'(c)$ and $m'(c)$ leads utility at r to change by

$$U_c + U_l + U_m m'(c)$$

which is strictly positive since by assumption at r , $-U_l < U_c$, $U_m \geq 0$ and $m'(c) \geq 0$.

If $U_m = 0$ then by (19), $U_{mc} = U_{mm} = U_{ml} = 0$ and thus many values of m solves (36). For the resulting utility $U(c, m(c), l(c))$ it is irrelevant which value we choose. For concreteness, we choose m to be the smallest value for which $U_m(c, m, l(c)) = 0$. At such a point increasing c and thus $l'(c)$ leads utility at r to change by

$$U_c + U_l$$

which is also strictly positive at r . This establishes the contradiction. Q.E.D.

In the next proposition we consider utility functions which are additively separable in leisure so that

$$(38) \quad U(c, m, l) = U(c, m) + V(l).$$

We also assume that there is some date r at which the budget is not balanced in that

$$(39) \quad \tau_r c_r + (1 - Q_{r,r+1})m_t \neq g_r + b_{-1,r}.$$

where $Q_{r,r+1}$ and τ_r are given by (11) and (13). The left-side of (39) is the revenues raised at r through direct tax levy and seignorage while the right-side are the fiscal commitments at r , namely government spending and initial commitments. We then have

Proposition 2. Suppose the initial nominal claims are zero in each period so that (22) holds, the second order conditions (35) hold, and there is some date s at which the budget is

not balanced, so that (39) holds. Then the date 0 Ramsey problem is time consistent only if the Friedman rule holds for each date.

Proof. We prove this by showing that if the Friedman rule does not hold at some date s the Ramsey problem is not time consistent. By way of contradiction we suppose that the Friedman rule does not hold at s but the Ramsey problem is time consistent. We show that this implies that the Ramsey problem must have a balanced budget at each date which contradicts (39).

Note that if the Friedman rule does not hold at s then $U_{ms} > 0$. Letting λ_0 and λ_1 denote the multipliers on the implementability constraints for the Ramsey problems at date 0 and date 1, we first show that $\lambda_0 = \lambda_1$. To see this consider the first order conditions to these problems for date s . From (22) we know that

$$(40) \quad 0 = \sum_{t=s}^{\infty} Q_{0,t} B_{-1,t}$$

Notice that when taking the first order conditions for m_s the terms of the form,

$$Q_{0s} \frac{\partial Q_{s,s+1}}{\partial m_s} \sum_{t=s}^{\infty} Q_{s+1,t} B_{-1,t}$$

equal zero in the date 0 Ramsey problem where $Q_{s,s+1} = 1 + U_{ms}/U_{ls}$ and the first order condition for m_s has the form of (28). Since $U_{ms} > 0$ it follows that

$$(41) \quad (1 + \lambda_0) + \lambda_0 \left[\frac{c_s U_{cms} + m_s U_{mms}}{U_{ms}} \right] = 0.$$

For the date 1 Ramsey problem we know that

$$0 = \sum_{t=s}^{\infty} Q_{1,t} B_{0,t}.$$

from Lemma 2 and the analogous terms are zero in the first order conditions for m_s in this problem. Thus, these first order conditions have the form of (28) also with λ_1 replacing λ_0 and thus can be written

$$(42) \quad (1 + \lambda_1) + \lambda_1 \left[\frac{c_s U_{cms} + m_s U_{mms}}{U_{ms}} \right] = 0.$$

By assumption, the Ramsey problem is time consistent so that the allocations in brackets in (41) and (42) are equal. Hence $\lambda_0 = \lambda_1$.

We next show that $\lambda_0 = \lambda_1$ implies that $b_{-1,t} = b_{0,t}$. Consider taking the first order conditions with respect to c_t and l_t in the date 1 Ramsey problem. We claim that at any date t the terms of the form

$$(43) \quad Q_{0t} \frac{\partial Q_{t,t+1}}{\partial c_t} \sum_{v=t+1}^{\infty} Q_{t+1,v} B_{-1,v} = 0$$

Suppose first that t is some date at which the Friedman rule holds. Then $U_{mt} = 0$ and from (19) it follows that $\partial Q_{t,t+1} / \partial c_t = (U_{mct} U_{lt} - U_{lct} U_{mt}) / U_{lt}^2 = 0$ where we have used $Q_{t,t+1} = 1 + U_{mt} / U_{lt}$. A similar argument implies $\partial Q_{t,t+1} / \partial l_t = 0$ and hence the corresponding terms are also zero when taking the first order conditions for l_t . If t is a date, like date s , at which the Friedman rule does not hold then (40) holds and these terms are zero as well.

Letting $R(c, m, l) = cU_c + mU_m + lU_l$ then using (43) in these first order conditions and then combining them gives that

$$(44) \quad U_{ct} + U_{lt} + \lambda_0 (R_{ct} + R_{lt}) = -\lambda_0 (U_{clt} + U_{llt}) b_{-1,t}$$

in the date 0 Ramsey problem and

$$(45) \quad U_{ct} + U_{lt} + \lambda_1 (R_{ct} + R_{lt}) = -\lambda_1 (U_{clt} + U_{llt}) b_{0,t}$$

in the date 1 Ramsey problem. Since $\lambda_0 = \lambda_1$, (44) and (45) imply that $b_{-1,t} = b_{0,t}$ for all t .

We now show that the budget must be balanced at each date so that (39) is contradicted. Assuming that a solution to the Ramsey problem at date 1 exists, the first-order condition for p_1 implies that

$$(46) \quad \frac{\lambda_1}{p_1} [M_0 + \sum_{t=1}^{\infty} Q_{1,t} B_{0,t}] = 0$$

Clearly, $\lambda_1 \neq 0$ and hence the real value of initial nominal liabilities is zero. (If $\lambda_1 = 0$ then the lump-sum allocations solve the Ramsey problem from date 1 on. But these allocations can be the continuation of the date 0 Ramsey allocations only if $\lambda_0 = 0$ as well, so that the lump-sum allocations solve the date 0 problem which they cannot given our assumption on initial conditions.) Suppose then that the real value of initial nominal liabilities is zero and evaluate the implementability constraint at this optimum. This gives

$$(47) \quad \sum_{t=1}^{\infty} \beta^{t-1} [c_t U_{c_t} + m_t U_{m_t} + l_t U_{l_t}] = - \sum_{t=1}^{\infty} \beta^{t-1} U_{l_t} b_{0,t}$$

Subtracting β times the date 1 implementability constraint from the date 0 implementability constraint gives

$$(48) \quad U_c c_0 + m_0 U_{m_0} + U_l l_0 = -U_l b_{-1,0}.$$

Using the resource constraint and the consumer first order conditions, (48) implies that the budget must be balanced at date 0

$$(49) \quad \tau_0 c_0 + (1 - Q_{0,1}) m_0 = g_0 + b_{-1,0}.$$

Repeating this argument for dates t and $t + 1$ gives that the budget must be balanced at every date t . This contradicts (39). *Q.E.D.*

We next show that if the Friedman rule holds in each period then the Ramsey problem is time consistent.

Proposition 3. Assume that the initial nominal claims are zero period by period so that (22) holds. If the Friedman rule holds for each period the Ramsey problem at date 0 is time consistent.

Proof. We begin by showing that the Ramsey problem for date 0 is time consistent for date 1 by constructing the appropriate initial conditions for the date 1 Ramsey problem, namely M_0 , ${}_0B$ and ${}_0b$. For the nominal assets we set $M_0 + B_{0,0} = 0$ and $B_{0,t} = 0$ for $t \geq 1$. (The breakdown of M_0 and $B_{0,0}$ is irrelevant as long as $M_0 > 0$.)

We construct the value for ${}_0b$ in a manner similar to that used by Lucas and Stokey (1983). Consider the first order conditions for m_t in the two problems, namely

$$(50) \quad U_{mt} + \lambda_s R_{mt} = -\lambda_s U_{lmt} b_{-1,t}$$

for $s = 0, 1$ where for the date 1 problem we have used our setting of the nominal claims and separability in labor. Since $U_{mt} = 0$ everywhere it follows from (19) that $R_{mt} = U_{mt} + c_t U_{cmt} + m_t U_{mmt} + l_t U_{mlt} = 0$. These first order conditions then trivially hold since the both sides of (50) are identically zero regardless of the multiplier. To derive a formula for ${}_0b$ consider the combined first order conditions for c_t and l_t , as in (44) and (45). Under the separability assumption these become

$$U_{ct} + U_{lt} + \lambda_0(R_{ct} + R_{lt}) = -\lambda_0(U_{lct} + U_{lnt})b_{-1,t}$$

in the date 0 Ramsey problem and

$$U_{ct} + U_{lt} + \lambda_1(R_{ct} + R_{lt}) = -\lambda_1(U_{lct} + U_{lnt})b_{0,t}$$

in the date 1 Ramsey problem. Subtracting these from each other and rearranging gives the formula for b_0 , namely

$$\lambda_1 b_{0,t} = \lambda_0 b_{-1,t} + \frac{(\lambda_1 - \lambda_0)(R_{ct} + R_{lt})}{U_{lct} + U_{llt}}.$$

Substituting this expression into the implementability constraint at $t = 1$ gives that λ_1 is implicitly defined by

$$\sum_{t=1}^{\infty} \beta^t R_t = - \sum_{t=1}^{\infty} \beta^t U_{lt} \left(\frac{\lambda_0}{\lambda_1} b_{-1,t} - \frac{(\lambda_1 - \lambda_0)(R_{ct} + R_{lt})}{\lambda_1(U_{lct} + U_{llt})} \right).$$

With the M_0 , B_0 and b_0 set in this manner the continuation of the date 0 Ramsey allocations solves the date 1 Ramsey problem and thus the Ramsey problem at date 0 is time consistent for date 1. Repeating this same argument for any date t gives our result. Q.E.D.

In this proposition we have chosen to make the restructured nominal claims have value zero period by period as in (22). It is worth pointing out that there are many other nominal debt sequences that can be used to render the Ramsey problem time consistent. Consider nominal debt sequences that satisfy the following

$$(51) \quad M_0 + \sum_{t=0}^{\infty} B_{0,t} = 0$$

so that at the Friedman rule the value of the initial nominal liabilities of the government of date 1 are zero and the value of nominal liabilities for any date $s \geq 1$ are negative in that

$$(52) \quad \sum_{t=s}^{\infty} B_{0,t} \leq 0.$$

We refer to condition (52) as requiring *negative tails* of the nominal liabilities inherited at date 1. We have

Lemma 3. If the Friedman rule is optimal in an economy in which initial nominal claims are zero period by period so that (22) holds then it is also optimal in an economy in

which these claims have a present value of zero at the Friedman rule and have negative tails, so that (51) and (52) hold.

Briefly, if the Friedman rule holds then any deviation from it has to lower some $Q_{t,t+1}$. Under the negative tails condition such a deviation puts smaller weight on the negative tail and hence raises the present value of initial nominal claims. This deviation does not improve utility and the Friedman rule is optimal.

More formally, consider a relaxed Ramsey problem in which the implementability constraint is written as an inequality and instead of having the bond prices connected to the allocations through $Q_{1,t} = \prod_{s=1}^{t-1} (1 + U_{ms}/U_{ls})$ we simply add the variables $Q_{t,t+1}$ for all t as new extra choice variables that must only satisfy $0 \leq Q_{t,t+1} \leq 1$ and $Q_{0,t} = Q_{0,1} \cdots Q_{t-1,t}$. It is clearly optimal to set $Q_{t,t+1}$ to minimize the present value of nominal claims. Under the negative tails assumption and (51), the lower bound for this present value is zero and is attained at $Q_{t,t+1} = 1$. Substituting in this value for $Q_{t,t+1}$ we see that the remaining problem is the same as one for the economy with zero nominal claims in each period, in which the Friedman rule is optimal. Thus, the solution to the relaxed problem is feasible for the original unrelaxed problem and hence is the solution.

Thus, in order to ensure time consistency there is a unique restructuring of the real debt but a whole variety of ways to restructure the nominal debt.

Combining Propositions 1 and 3 gives the following proposition.

Proposition 4. If the utility function is separable and homothetic so that (27) holds and initial nominal claims are zero period by period so that (22) holds then Ramsey problem at date 0 is time consistent.

Clearly, our results, especially Proposition 2, are at odds with the results in Persson,

Persson and Svensson (1987). In their analysis they constructed a nominal debt sequences to be inherited by the date 1 government and supposed that taking this sequence as an initial condition the date 1 government chose an interior point for p_1 , so that $0 < p_1 < \infty$. As Lemma 2 shows, unless the Friedman rule is satisfied, the government will choose the lower corner. Thus, there are restrictions on the nominal debt sequence they did not take into account.

3. Sustainable plans

Here we relate the Lucas-Stokey notion of time consistency to the literature on sustainable plans and credible policy. We show that if the Ramsey equilibrium is time consistent then it is sustainable. More precisely, we show that the Ramsey allocations and policies are sustainable outcomes generated by a Markov sustainable equilibrium. It is worth noting that the converse is clearly not true, that is sustainable outcomes are not typically time consistent.

In the Lucas-Stokey definition the government at date 0 solves a problem under the presumption that it has the ability to commit to all its future policies and that consumers act under this presumption as well. What the government at date 0 actually gets to set, however, are the date 0 policies including the new initial conditions for the government at date 1. The problem at date 0 is time consistent for the problem at date 1 if there exists initial conditions such that the government at date 1, under a similar presumptions about commitment chooses to continue with the allocations and policies chosen by the government at date 0. In this definition the government at date 0 does not explicitly think through how altering the initial conditions for the government of date 1 affects its choices, since the government at date 0 simply presumes it can commit all the future policies.

The sustainable plan literature takes lack of commitment as given and explicitly builds it into the definition of an equilibrium. In this definition the government at date 0 realizes both that it cannot commit to all its future policies and that consumers realize that as well. This government also explicitly thinks through how altering the initial conditions for the date 1 problem affects the choices of the date 1 government.

In this literature, the lack of commitment is modeled by having the government choose policy sequentially¹. Consumer allocations, prices and government policy are specified as functions of the history of past policies of the government. These functions specify behavior for any possible history, even for those in which the government deviates from prescribed behavior. In contrast, the time consistent equilibrium simply specifies a given sequence of allocations, prices and policies and is thus not directly comparable to a sustainable equilibrium. Along the equilibrium path, however, a sustainable equilibrium generates a particular sequence of allocations, prices and policies, called a *sustainable outcome* which is comparable to the sequences specified by a time consistent equilibrium.

For a version of the Lucas and Stokey (1983) economy without money, Chari and Kehoe (1993) show that the sustainable outcome generated by a Markov sustainable equilibrium solves a simple programming problem. With a little work one can extend their results to our economy and show that for some given initial conditions M_{-1} , b_{-1} , and B_{-1} , the allocations $({}_0c, {}_0m, {}_0l)$ are *sustainable Markov allocations* if and only if they are part of the solution to the following programming problem: choose allocations $({}_0c, {}_0m, {}_0l)$, nominal money supplies M_1 , real and nominal debt b_1 and B_1 to solve the *sustainable Markov problem*

$$(53) \quad V_0(M_{-1}, b_{-1}, B_{-1}) = \max \sum_{t=0}^{\infty} \beta^t U(c_t, m_t, l_t)$$

subject to the resource constraint for $t \geq 0$, the implementability constraint for all $t = 0$

$$(54) \quad \sum_{t=0}^{\infty} \beta^t [c_t U_{ct} + m_t U_{mt} + l_t U_{lt}] = -\frac{U_{l0}}{p_0} [M_{-1} + \sum_{t=0}^{\infty} Q_{0,t} B_{-1,t}] - \sum_{t=0}^{\infty} \beta^t U_{lt} b_{-1,t}.$$

and $t = 1$ which is given by

$$(55) \quad \sum_{t=1}^{\infty} \beta^{t-1} [c_t U_{ct} + m_t U_{mt} + l_t U_{lt}] = -\frac{U_{l1}}{p_1} [M_0 + \sum_{t=1}^{\infty} Q_{1,t} B_{0,t}] - \sum_{t=1}^{\infty} \beta^{t-1} U_{lt} b_{0,t}.$$

and the *sustainability constraint* for all $t > 0$

$$(56) \quad \sum_{t=1}^{\infty} \beta^{t-1} U(c_t, m_t, l_t) \geq V_1(M_0, b_0, B_0)$$

where $Q_{0,t} = \prod_{s=0}^{t-1} (1 + U_{ms}/U_{ls})$ and $p_1 = M_1/m_1$ and the functions $V_t(M_{t-1}, b_{t-1}, B_{t-1})$ are defined recursively using (53). The sustainability constraint captures the restriction that whatever sequence of allocations from date 0 to infinity is contemplated by the government at date 0, given the state variables (M_0, b_1, B_1) that this government passes to the government at date 1, the government at date 1 has an incentive to implement the continuation of these allocations from date 1. The government at date 1 faces a similar constraint with respect to the government at date 2 and so on for the government at all future dates. (One subtle point is that here we are defining a equilibrium that is utility-Markov, in that the payoffs depend only on the state variables, rather than the more common strategy-Markov in that the allocations and associated policies depend only on the state variables. We do so to avoid the problems raised by Bernheim and Ray with respect to strategy-Markov equilibria. For details see Chari and Kehoe (1993)).

Notice the *sustainable Markov problem* is essentially the Ramsey problem at date 0 with two extra constraints the implementability constraint at date 1 and the sustainability constraint and extra choice variables (M_0, b_1, B_1) and p_1 . The following is then immediate.

Proposition 5. If the utility function is separable and homothetic so that (27) holds and initial nominal claims are zero period by period so that (22) holds then Ramsey allocations are Markov sustainable allocations.

Our previous propositions contain the essence of the proof. Letting $V_0^R(M_{-1}, b_{-1}, B_{-1})$ denote the value of the Ramsey problem at date 0 with state variables (M_{-1}, b_{-1}, B_{-1}) . Since the Ramsey problem is a less constrained version of the sustainable Markov problem its value is necessarily higher, so

$$V_0^R(M_{-1}, b_{-1}, B_{-1}) \geq V_0(M_{-1}, b_{-1}, B_{-1}).$$

Thus, if the Ramsey allocations are feasible for the sustainable Markov problem they necessarily solve it. Consider then the Ramsey allocations given the state variables (M_{-1}, b_{-1}, B_{-1}) . These allocations clearly satisfy the resource constraint and the implementability constraint at date 0 in the sustainable Markov problem. Given the values for the new state variables (M_0, b_0, B_0) constructed in the proof of Proposition 3, these state variables plus the continuation of the date 0 allocations clearly satisfy the remaining constraints of the sustainable Markov problem, namely the implementability constraint at date 1 and the sustainability constraint.

4. The analogy with a real economy

In this section we develop the connection between our results for a monetary economy and related results for a real economy. In our monetary economy we showed that the incentive to inflate away nominal debt severely restricted the government at any date from using nominal debt to influence the behavior of its successors. In effect the government was endogenously forced to use only one type of debt, real debt to influence its successors.

Here we consider a real economy with two consumption goods and two consumption taxes. We suppose that the government has access to only one real bond of all maturities. In this environment the government at any date is exogenously forced to use only one type of debt, a single real bond, to influence the behavior of its successors. We show that the Ramsey policies are time consistent if and only if the optimal taxes are uniform in the sense that it is optimal to tax the two consumption goods at the same rate in all periods. We then show that the sufficient conditions for uniform consumption taxes in the real economy are similar to those for the Friedman rule in the monetary economy.

The resource constraint for this economy is

$$(57) \quad c_{1t} + c_{2t} + g_t = l_t$$

where c_{1t} and c_{2t} are the consumption goods, g_t is government consumption and l_t is labor.

The utility function is

$$(58) \quad \sum_{t=0}^{\infty} \beta^t U(c_{1t}, c_{2t}, l_t)$$

and the date 0 budget constraint is

$$(59) \quad \sum_{t=0}^{\infty} q_{0t} [(1 + \tau_{1t})c_{1t} + (1 + \tau_{2t})c_{2t} - l_t] = \sum_{t=0}^{\infty} q_{0t} b_{-1t}$$

where b_{-1} is the sequence of initial real debt of all maturities. The first order conditions are

$$(60) \quad \beta^t U_{it} = \theta q_{0t} (1 + \tau_{it})$$

$$(61) \quad -\beta^t U_{lt} = \theta q_{0t}$$

for $i = 1, 2$ and for all t , where U_{it} denotes the derivative with respect to c_{it} . Notice that the consumption taxes are *uniform* in the sense that $\tau_{1t} = \tau_{2t}$ for all t if and only if

$$(62) \quad U_{1t} = U_{2t}$$

for all t . The implementability constraint for this economy is

$$(63) \quad \sum_{t=0}^{\infty} \beta^t R(c_{1t}, c_{2t}, l_t) = - \sum_{t=0}^{\infty} \beta^t U_{1t} b_{-1t}$$

where $R(c_1, c_2, l) = c_1 U_1 + c_2 U_2 + l U_l$.

The Ramsey problem at date 0 is to maximize (58) subject to (57) and (63) given a initial real debt b_{-1} . Time consistency is defined in the obvious fashion. We then have the following proposition.

Proposition 5. Assume the utility function is additively separable in leisure and there is some period for which the budget is not balanced under the Ramsey policies. Then the Ramsey problem is time consistent if and only if consumption taxes are uniform.

Proof. Suppose first that the consumption taxes are uniform, we show that the Ramsey problem is time consistent. The first order conditions for the Ramsey problem at date 0 are

$$(64) \quad U_{it} + \lambda_0 R_{it} = \gamma_{0t}$$

$$(65) \quad U_{lt} + \lambda_0 (R_{lt} + U_{l1t} b_{-1,t}) = -\gamma_{0t}$$

where λ_0 and $\beta^t \gamma_{0t}$ are the multipliers on the implementability constraint and the resource constraint. Notice from (64) that $U_{1t} = U_{2t}$ implies that

$$(66) \quad R_{1t} = R_{2t}$$

Adding the first order conditions (64) to (65) gives

$$(67) \quad U_{it} + U_{lt} + \lambda_0 (R_{it} + R_{lt} + U_{l1t} b_{-1,t}) = 0$$

for $i = 1, 2$ and $t \geq 0$. For the Ramsey problem at date 1, given some inherited debt b_0 , the corresponding first order conditions are

$$(68) \quad U_{it} + U_{lt} + \lambda_1 (R_{it} + R_{lt} + U_{l1t} b_{0,t}) = 0$$

for $i = 1, 2$ and $t \geq 1$. Since taxes are uniform (62)) holds for all t and the two equations in (67) are the same equations and likewise for the two equations in (68). Subtracting (68) from (67) for $t \geq 1$, gives

$$(69) \quad \lambda_1 U_{1t} b_{0t} = \lambda_0 U_{1t} b_{-1t} + (\lambda_0 - \lambda_1)(R_{1t} + R_{2t})$$

Equation (69) gives the formula for the debt that the government at date 0 passes on to the government at date 1. The multiplier λ_1 can be found by substituting (69) into the implementability constraint. Given this debt b_0 and the multiplier λ_1 the solution to the Ramsey problem at date 1 is the continuation of the solution to the Ramsey problem at date 0.

Suppose next that the consumption taxes are not uniform, so that there is some date s at which, say, $U_{1s} > U_{2s}$. We will show that if the Ramsey plan is time consistent the budget must be balanced in each period. Subtracting the two equations in (67) from each other gives

$$(70) \quad U_{1t} - U_{2t} + \lambda_0(R_{1t} - R_{2t}) = 0$$

and doing the same for the two equations in (68) gives

$$(71) \quad U_{1t} - U_{2t} + \lambda_1(R_{1t} - R_{2t}) = 0.$$

Since $U_{1s} > U_{2s}$ and $\lambda_0 > 0$ then (70) implies that $R_{1s} < R_{2s}$. Evaluating (70) and (71) at $t = s$ gives that $\lambda_0 = \lambda_1$. From (69) we note that $b_{0t} = b_{-1t}$ for $t \geq 0$ and hence we can write the implementability constraint at date 1 as Given this fact we can subtract β times the implementability constraint at date 1, namely,

$$(72) \quad \sum_{t=1}^{\infty} \beta^{t-1} R(c_{1t}, c_{2t}, l_t) = - \sum_{t=1}^{\infty} \beta^{t-1} U_{1t} b_{-1t}.$$

Subtracting β times the implementability constraint at date 1 from the implementability constraint at date 0 gives

$$R(c_{10}, c_{20}, l_0) = U_0 b_{-10}$$

which, using the resource constraint (57) and the consumer first order conditions (60) and (61) can be written

$$\tau_{10}c_{10} + \tau_{20}c_{20} = g_0 + b_{-10}$$

so that the government's budget is balanced at date 0. A similar argument shows that the government's budget is balanced at each date t . *Q.E.D.*

It is easy to use the logic developed in Proposition 1 to show that if the utility function U is additively separable and homothetic in that

$$(73) \quad U(c_1, c_2, l) = w(c_1, c_2) + V(l)$$

with w homothetic, then uniform consumption taxes are optimal. Notice that this condition is similar to the sufficient condition for the Friedman rule in Proposition 1. It is also precisely the obvious sufficient condition for the Friedman rule to be optimal to the cash-credit monetary economy that is analogous to the real economy considered here.

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