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**On Inflation and the Persistence of Shocks
to Output**



On Inflation and the Persistence of Shocks to Output

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Abstract

This paper investigates the possibility that the effects of shocks to output depend on the level of inflation. The empirical analysis extends Elwood's (1998) framework by incorporating an inflation threshold process in the examination of the stochastic properties of output. This extension allows us to estimate the level of inflation at which the behavior of shocks to output change. The results indicate that shocks to output indeed have asymmetric effects depending on the level of inflation. In particular, negative shocks are more detrimental when inflation is high, while positive shocks are more persistent when inflation is low.

JEL Classification: E31, E32, E52, E58

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1 Introduction

The purpose of this paper is to examine whether the level of inflation matters for the persistence of output growth. The idea that inflation could have such threshold effects is worth investigating because some authors have suggested that a low inflation environment was instrumental in generating the unprecedented strong and sustained output growth rates recently witnessed in some countries. For instance, Taylor (1998) has pointed out that the United States experienced its two longest post-war expansions after 1983 and has suggested that monetary policy was the main cause of it. A quotation from him states that “... by keeping the inflation low and stable ... the Fed (Federal Reserve) has succeeded in stabilizing the economy and making recessions less frequent, smaller, and shorter.”

We also know, from a separate literature, that shocks may have asymmetric effects on the persistence of output growth. For instance, Beaudry and Koop (1993) showed that when an index variable capturing the depth of recessions is included in a standard ARMA model, positive shocks will generate substantially different output dynamics than will negative shocks. Needless to say, if the effects of shocks are indeed asymmetric, imposing symmetry will bias the estimates of the persistence parameters.

In this paper we account for the possibility that both inflation and own shocks influence output growth behavior and that this impact can vary with the level of the first and the sign of the second. The model that we propose for this is a generalization of the unobserved-components threshold framework suggested by Elwood (1998), which initially had been used to examine the asymmetric effect of own disturbances on output growth. More precisely, the disturbance term was treated as an unobserved component and its estimated sign determined the regime within which output growth would evolve.

A main advantage of Elwood’s methodology over Beaudry and Koop is that, instead of using a possibly imperfect proxy, shocks are allowed to directly influence

output growth behavior². On the other hand, Beaudry and Koop use ARMA specifications, which are more flexible than Elwood's low-order AR or MA models. These observations suggest that a more general framework, combining elements from both studies, could prove a useful modeling alternative.

Working along these lines, we retain the main Elwood structure but extend it to an ARMA setting. In addition, to integrate the possible role of the inflation environment on output growth, we further generalize this model by allowing for multiple threshold effects. Consequently, the parameters of output growth are permitted to change depending on (1) whether disturbances are positive or negative, and (2) on whether inflation is above or below some threshold level. We then test for these distinct effects using Canadian data, since Canada has had an announced low-inflation policy from the early nineties. The estimation is carried out using maximum likelihood and hypotheses are tested with the aid of Hansen's (1996) bootstrap test procedure for when a nuisance parameter is present only under the alternative.

Our results concur with the conclusions of Beaudry and Koop in finding that shocks indeed have asymmetric effects on output. However we further show that the inflation environment at the time of shock is crucial to seeing which of these shocks has the higher persistence. Thus, under low inflation, a positive shock is found to be more persistent than a negative shock of the same size. On the other hand, the situation is reversed when inflation is above its threshold value. Therefore, low inflation is associated with a healthier output growth dynamics compared to an environment of high inflation. Finally, the estimated threshold value for Canada is approximately 4.5%. This result is interesting considering that the upper limit of the announced inflation target bands by the Bank of Canada has never exceeded this value.

²This might be partly the reason why, for the same data, Elwood did not find significantly different effects.

Section 2 presents our generalized multiple-threshold framework and reports the maximum likelihood estimation results. Next, in section 3, we test for the threshold effect using the likelihood ratio statistic. Since the threshold parameter is not identified under the null, the asymptotic distribution of the test statistic is simulated using Hansen’s (1996) procedure. The test results indicate that the threshold effect is indeed significant. Section 4 describes our findings graphically, that is impulse-response functions show the effects of one-time positive and negative shocks to output growth and level. Finally, section 5 concludes.

2 The Threshold ARMA model

The approach that we adopt, to investigate the possibility of a threshold effect in the inflation-output relation, is based on the class of threshold autoregressive models introduced by Tong (1978). In such models, switches in the parameter values are endogenously generated by a fixed lag of the *observed* series. See Tong and Lim (1980) and Tong (1983, 1990) for more details.

Elwood (1998) proposed an extension to threshold autoregressive and moving average models where the parameters switch depending on the sign of a fixed lag of the *unobserved* shocks to the series. His unobserved components methodology provides a framework for detecting the presence of asymmetries in the persistence of shocks to output. In order to investigate the additional threshold effects of inflation, we consider the following four-regime threshold autoregressive moving average model:

$$\Delta y_t = \mu + \phi_1^p(\Delta y_{t-1} - \mu) + \varepsilon_t + \theta_1^p \varepsilon_{t-1}, \quad \varepsilon_{t-1} \geq 0, \quad \pi_{t-1} \geq d, \quad (1)$$

$$\Delta y_t = \mu + \phi_1^n(\Delta y_{t-1} - \mu) + \varepsilon_t + \theta_1^n \varepsilon_{t-1}, \quad \varepsilon_{t-1} < 0, \quad \pi_{t-1} \geq d, \quad (2)$$

$$\Delta y_t = \mu + \phi_2^p(\Delta y_{t-1} - \mu) + \varepsilon_t + \theta_2^p \varepsilon_{t-1}, \quad \varepsilon_{t-1} \geq 0, \quad \pi_{t-1} < d, \quad (3)$$

$$\Delta y_t = \mu + \phi_2^n(\Delta y_{t-1} - \mu) + \varepsilon_t + \theta_2^n \varepsilon_{t-1}, \quad \varepsilon_{t-1} < 0, \quad \pi_{t-1} < d, \quad (4)$$

where Δy_t is output growth, π_t is the rate of inflation and $\varepsilon_t \sim i.i.d. N(0, \sigma^2)$. Given this specification, output growth Δy_{t+1} depends on the sign of the time t shock ε_t and the level of inflation π_t . Therefore, the persistence of positive and negative shocks may be different depending on the values of the parameters ϕ_j^i , θ_j^i , $j = 1, 2$, $i = p, n$. For example, if $\phi_2^p + \theta_2^p > \phi_2^n + \theta_2^n$ and $\phi_1^p + \theta_1^p < \phi_1^n + \theta_1^n$, then positive shocks have more persistent effects on output growth than do negative shocks when inflation is low, and vice versa when inflation is high.

For a given value of the inflation threshold level d , the remaining model parameters can be estimated using the modified Kalman filter proposed by Elwood (1998). The appendix provides the state-space representation of the threshold ARMA(1,1) model described above with a description of the estimation methodology. Denote by $\hat{L}_1(d)$ the maximized likelihood function for a given value of d . The estimate for d can then be defined as

$$\hat{d} = \arg \max_{d \in \mathcal{D}} \hat{L}_1(d), \quad (5)$$

where \mathcal{D} is the set of admissible values for d . The range of admissible values for the inflation threshold parameter was defined as the observed range of inflation levels, with 15% trimmed at both ends.³

Note that, in principle, higher-order threshold ARMA models could be considered by extending the estimation methodology. However, a drawback is that the number of parameters grows exponentially with the number of regimes. Given that in this paper our goal is to simply investigate whether the data is supportive of the presence of threshold effects, we limit our analysis to the first-order case. Nevertheless, despite its apparent simplicity, the threshold ARMA(1,1) is a parsimonious representation of a potentially highly asymmetric time series. To see this, note that the proposed model can be decomposed into four unobserved components, each receiving all the shocks that are specific to their sign and the

³The trimming rule follows Andrews (1993) and Hansen (1996).

pertaining inflation regime:

$$\Delta y_t = \mu + \Delta y_{t1}^p + \Delta y_{t1}^n + \Delta y_{t2}^p + \Delta y_{t2}^n + \varepsilon_t, \quad (6)$$

where Δy_{tj}^i are infinite moving-average processes for $j = 1, 2$ and $i = p, n$. Thus, for example, the first unobserved component is the infinite sum of all past positive shocks that occurred while inflation was high:

$$\Delta y_{t1}^p = \frac{\theta_1^p B}{1 - \phi_1^p B} \varepsilon_t, \text{ for all } \varepsilon_{t-k} \geq 0, \pi_{t-k} \geq d, \quad (7)$$

defined for $k \geq 1$ and where B is the lag operator such that $B^k z_t = z_{t-k}$. The others components in (6) are defined in a similar fashion.⁴

The proposed model in equations (1) through (4) was estimated using Canadian data on real GDP and on the associated implicit prices, over the period 1965:Q1 to 2000:Q3. Specifically, the growth rate of output is the annualized log difference of seasonally adjusted real GDP while inflation is the annualized log difference of the GDP deflator. Trimming 15% of the highest and lowest values of the inflation series yielded the interval [1.5,7.5] for \mathcal{D} , over which we defined a grid of 60 possible values that vary by increments of 0.1. The value of the inflation threshold parameter estimated by the grid-search method resulted in $\hat{d} = 4.40$, which is statistically significant as will be seen in the next section. The estimation results for the remainder of the model parameters are summarized in Table 1, while Figure 1 shows the estimated inflation threshold level against the output and inflation series.

2.1 Diagnostic checks

The quasi-maximum-likelihood parameter estimates presented in Table 1 were used in the modified Kalman filter to obtain residuals, e_t (see equation (31) in the

⁴Seen this way, the proposed model is in fact an extension of the asymmetric moving average model originally developed in Wecker (1981), as were the models proposed by Elwood (1998).

Table 1
Estimation Results

μ	ϕ_1^p	ϕ_2^p	θ_1^p	θ_2^p	ϕ_1^n	ϕ_2^n	θ_1^n	θ_2^n	σ
4.13 (0.04)	0.86 (0.22)	-0.56 (0.005)	-0.99 (0.06)	0.99 (0.09)	-0.31 (0.006)	0.96 (0.71)	0.98 (0.07)	-0.99 (0.06)	3.31 (0.03)

Note: Superscripts p refer to coefficients of positive shocks, superscripts n refer to coefficients of negative shocks, subscripts 1 refer to a high inflation regime while subscripts 2 indicate coefficients estimated for the low inflation case. The numbers in parenthesis are the asymptotic standard errors derived from the elements along the main diagonal of the inverse of the information matrix.

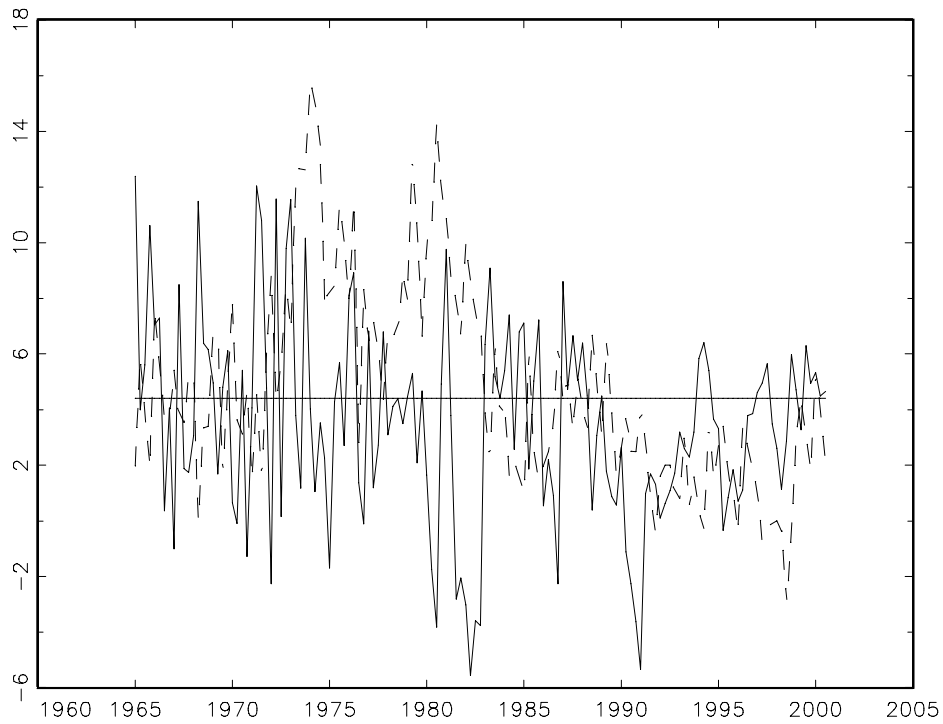


Figure 1: Growth rates of Canadian GDP (solid line) and the GDP deflator (dashed line). The horizontal line represents the estimated inflation threshold level.

Table 2
Diagnostic Tests

z_N	mz_H	$z_{Q(1)}$	$z_{Q(2)}$	$z_{Q(3)}$	$z_{Q(4)}$	$z_{Q(5)}$	$z_{Q(6)}$
6.50*	10.78	0.45	2.87	5.90	8.69	8.82	10.52

Note: The statistic z_N is a test for residual non-normality, which asymptotically is χ_2^2 . The star indicates statistical significance at the 5% level. The statistic z_H is a heteroscedasticity test for which mz_H is asymptotically χ_m^2 , with $m = T/3$. Finally, $z_{Q(p)}$ are autocorrelation tests based on p lags, for $p = 1, \dots, 6$, that asymptotically are χ_p^2 .

appendix). Using this series, a number of statistics were then computed to test the model specification.

The statistic z_N , which asymptotically is χ_2^2 , is the Bowman and Shenton (1975) test for residual non-normality computed as

$$z_N = (T/6) \left(\hat{\sigma}^{-3} \sum_{t=1}^T e_t^3 / T \right)^2 + (T/24) \left(\hat{\sigma}^{-4} \sum_{t=1}^T e_t^4 / T - 3 \right)^2, \quad (8)$$

where $\hat{\sigma}^2 = \sum_{t=1}^T e_t^2 / T$. The computed value of 6.50 has an associated p-value of 0.0041, which is marginally significant at the conventional 5% level. The statistic z_H is a Goldfeld and Quandt (1965) type heteroscedasticity test computed as

$$z_H = \frac{\sum_{t=T-m+1}^T e_t^2}{\sum_{t=1}^m e_t^2}, \quad (9)$$

with $m = T/3$, and where mz_H is asymptotically χ_m^2 . Finally, the statistics $z_{Q(p)}$, for $p = 1, \dots, 6$, are Ljung and Box (1979) autocorrelation tests computed as

$$z_{Q(i)} = T(T+2) \sum_{h=1}^p (T-h)^{-1} \hat{\rho}(h)^2, \quad (10)$$

and distributed asymptotically as χ_p^2 , where $\hat{\rho}(h) = \sum_{t=h+1}^T e_t e_{t-h} / \sum_{t=1}^T e_t^2$ is the sample autocorrelation. See Harvey (1990, Section 5.2) for more on these tests. The results of these diagnostic checks are presented in Table 2. Besides the slight departure from normality, the residuals seem supportive of the model specification.

3 Testing for Threshold Effects

There are two hypotheses of interest related to threshold effects. The first is that there is no inflation threshold effect, i.e., the level of inflation has no effect on output growth. In the notation of the model, this hypothesis is represented as

$$H_{01} : \phi_1^i = \phi_2^i \text{ and } \theta_1^i = \theta_2^i, \quad \text{for } i = p, n. \quad (11)$$

The second hypothesis is that shocks of different signs have similar effects on output. That is,

$$H_{02} : \phi_j^p = \phi_j^n \text{ and } \theta_j^p = \theta_j^n, \quad \text{for } j = 1, 2. \quad (12)$$

Statistical testing of H_{02} can be performed on the basis of standard asymptotic theory. We thus compute the usual likelihood ratio statistic defined as $LR_{02}(d) = 2\log(\hat{L}_1(d)/\hat{L}_{02}(d))$, where we recall that $\hat{L}_1(d)$ is the maximized value of the unconstrained likelihood function, while $\hat{L}_{02}(d)$ is that under H_{02} . Under H_{02} , the likelihood ratio LR_{02} is asymptotically distributed as $\chi^2(r)$, with degrees of freedom $r = 4$. We find that $LR_{02}(\hat{d}) = 10.26$, which has a p-value of 0.0375, indicating that shocks of different signs have significantly different effects on output.

Statistical testing of H_{01} is not as straightforward. Under the null hypothesis of no inflation threshold effect, the threshold parameter d is not identified. In such a case, standard asymptotic inference is invalid since the information matrix is singular under the null hypothesis. To account for the fact that some parameters are present only under the alternative, we use the bootstrap test procedure developed in Hansen (1996). Hansen's procedure allows us to simulate the limiting distribution of the supremum likelihood ratio that results from a maximization over the space of the threshold parameter. The bootstrap critical values were based on 1000 replications of the simulation procedure.

Figure 2 plots the likelihood ratio statistic $LR_{01}(d) = 2\log(\hat{L}_1(d)/\hat{L}_{01}(d))$ for the 60 values of d , where \hat{L}_{01} is the maximized value of the likelihood function under

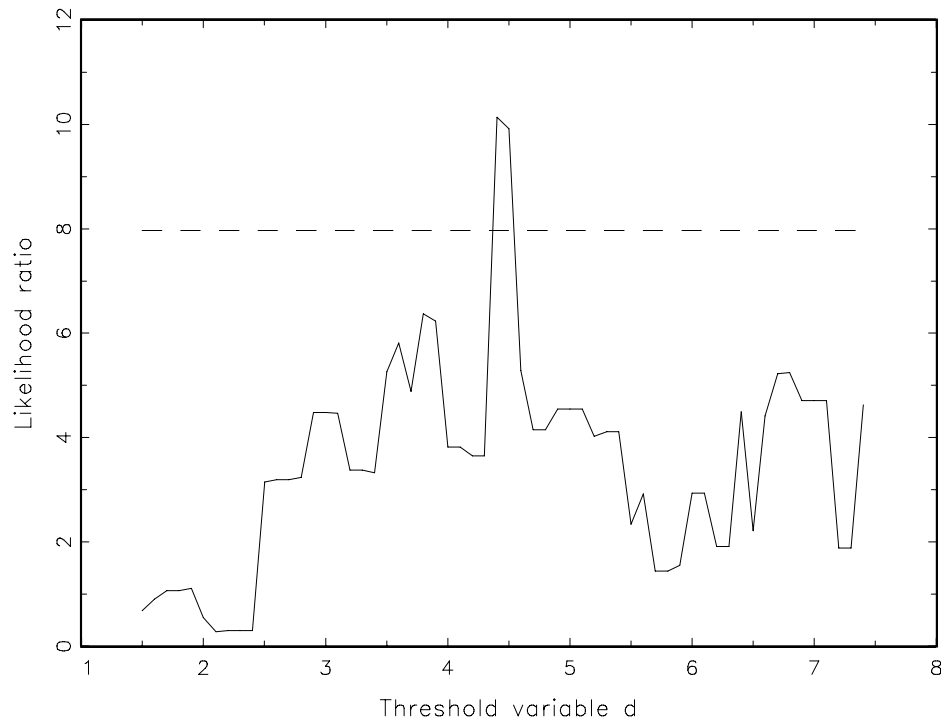


Figure 2: The solid line corresponds to the value of the likelihood function for a given value of the inflation threshold variable d . The dashed line is the bootstrap 5% critical value. Values for which the likelihood is above the dashed line yield the 95% confidence interval for d .

H_{01} . Obviously $LR_{01}(d)$ reaches its maximum value at the threshold estimate $\hat{d} = 4.40$. The dotted line in the graph represents the bootstrap critical value at the 5% level. Thus values for which the likelihood is above the dashed line yield the 95% confidence interval for d . We see from the graph that the confidence interval is quite tight around the threshold estimate. It appears that the behavior of output growth does indeed depend on the level of inflation.

4 Impulse Responses

In this section we describe impulse responses for the growth rate and the level of output. Impulse responses can be computed by taking the difference between a shocked and a base case. Since our model is nonlinear, these functions however depend both on the lagged values of output before the shock and on the size of the imputed shock. Following Beaudry and Koop (1993) we therefore construct unconditional impulse responses, in the sense that we compare the after-shock effect to a base case where output growth equals its mean and where all past disturbances are zero. In addition, we normalize the shocks to have a unit variance so that a shock of one unit is equivalent to a shock of one standard deviation. Therefore, the impulse response function of output growth to a shock v at time $t = 0$ for say, the high-inflation regime, τ periods ahead is given by

$$RF(\Delta y_\tau; v) = E[\Delta y_\tau | \Delta y_0 = \mu + v, \varepsilon_0 = v, \pi_\tau \geq d] - \mu, \quad (13)$$

defined for $\tau \geq 1$, and where we set $v = \sigma = 1$. The response function of the level of output is then obtained simply as

$$RF(y_\tau; v) = RF(\Delta y_\tau; v) + RF(y_{\tau-1}; v), \quad (14)$$

with $RF(y_0; v) = v$. Therefore, conditional on a given inflation regime, the impulse responses are independent of the history of the time series as in the case of a

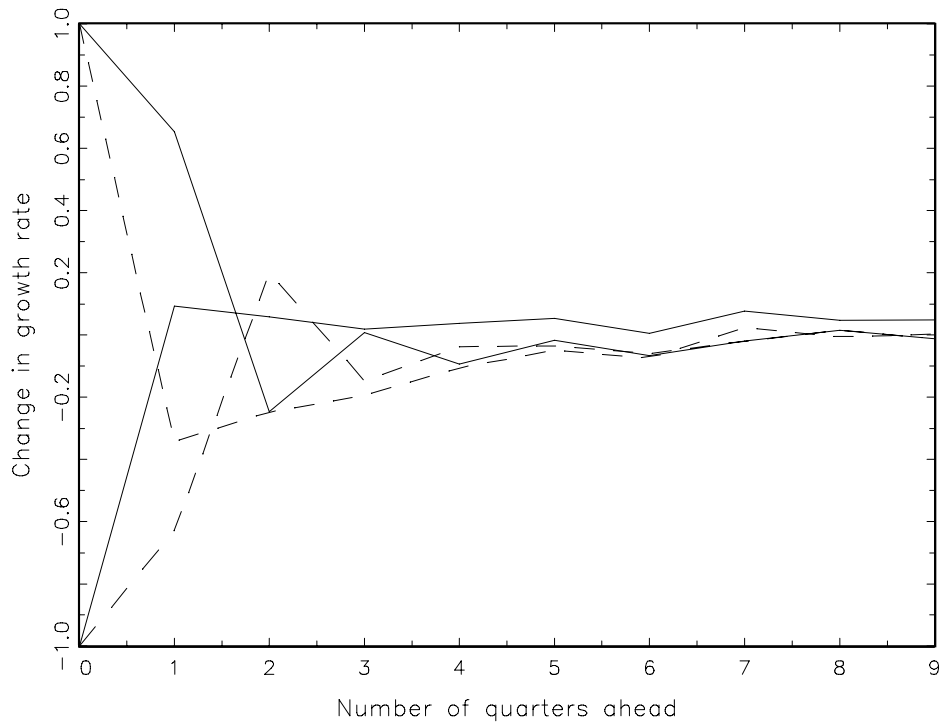


Figure 3: Impulse response of the growth rate of output. The solid lines are associated with the low inflation regime, while the dashed line are associated with the high inflation regime.

standard linear model. However, unlike the standard case, the impulse responses depend on the sign of the imputed shock.⁵

⁵See Potter (1995) for more on non-linear impulse response functions.

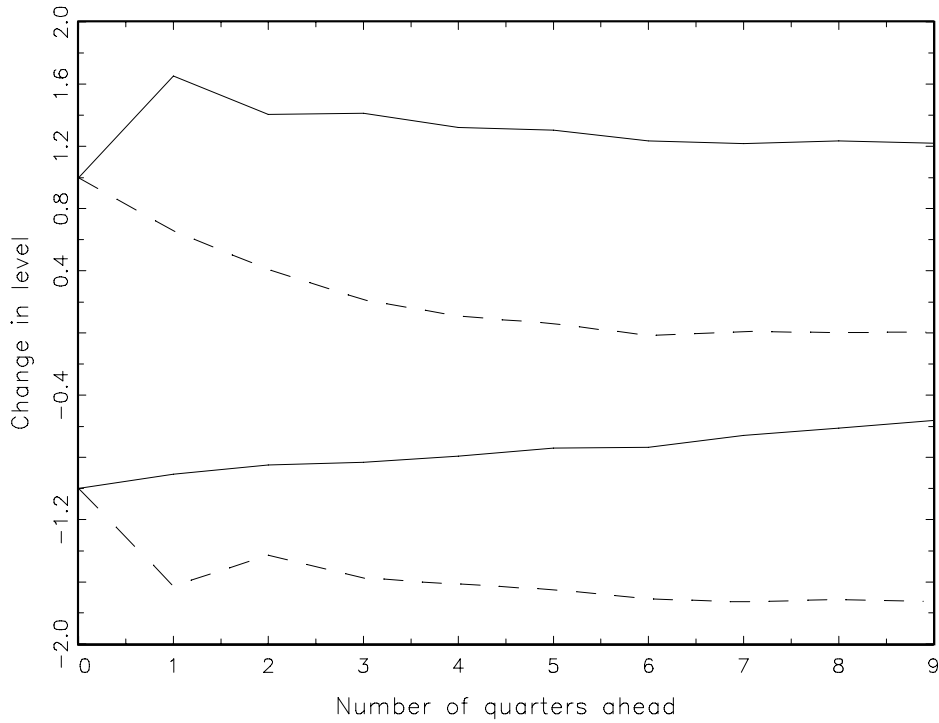


Figure 4: Impulse response of the level of output. The solid lines are associated with the low inflation regime, while the dashed line are associated with the high inflation regime.

Figures 3 and 4 show the impulse response functions for the growth rate and the level of output, respectively. The solid lines represent the responses to shocks when inflation is in the low regime (i.e. when inflation is below the threshold level), whereas the dashed lines are the corresponding responses in the high-inflation regime.

4.1 The Role of Inflation

Looking at the effect of a unit positive shock on output growth, we can see that the impact is twice as persistent when the economy is in the low inflation regime than when it is in the high one. In other words, the effect of this shock on output growth is felt over two quarters instead of just one. On the other hand, we find that a negative shock causes the growth rate to rebound twice as quickly to its mean level when inflation is in the low regime than when it is in the high one.

Thus, a negative shock is less persistent when the economy is experiencing low inflation compared to a situation of high inflation.

Turning now to the impulse responses on the output level, we find that a positive, one standard deviation shock, in a situation of low inflation, will cause output to increase substantially and to settle two years later at a level that is 20% higher than the amount of the shock. In contrast, the same positive shock, this time under a high inflation regime, will have dissipated within a year and a half. Clearly then, a positive shock has a much more beneficial effect on the economy when it arrives at a time of low inflation. The effect of a negative one-unit shock is equally telling. Under high inflation, such a shock causes output to decline considerably, such that two years later, output will have diminished by 1.5 times the amount of the shock. In contrast, with low inflation, the effect of the shock decreases albeit at a slow rate.

Based on these figures, we can thus conclude that a low-inflation regime is clearly much more desirable. Of course, it may not be so much that the low level of inflation itself is the cause of the good times described above, but rather that this variable is able to capture the presence of an underlying set of structural nonlinear conditions that are favorable to the economy. Note that, despite the fact that inflation here has been defined as the growth rate of the GDP deflator, it is interesting that, since its announcement, the upper limit of Canada's target range for CPI inflation has never exceeded the 4% level.

4.2 Asymmetry of Output Shocks

For a given inflation level, we now compare the response of the GDP to positive and negative shocks of equal size. Consider first the situation where inflation is above its threshold value (the dashed lines in Figure 4). It is easy to see that a positive standard-deviation shock to output is less persistent than an equivalent negative shock. In fact, while the former has already dissipated one and a half

years after the initial impact, the negative shock causes output to fall by an extra half a standard deviation immediately and commits GDP to that low level well beyond two years. Turning now to the reaction of output to these shocks in a low inflation environment (the solid lines in Figure 4), again, we find that there is asymmetry. Two years after a one-unit positive shock, output rises by an additional 20-25% of the amount of the shock. In contrast, the negative shock is not exacerbated, but slowly starts dissipating over time.

It can therefore be concluded that, for a given inflation regime, shocks to output have asymmetric effects. Furthermore, this asymmetry is more pronounced in the high inflation environment. Interestingly, while a negative shock has a worse size-effect under high inflation than a positive shock has under low inflation, both shocks nevertheless display similar dynamic propagations.

5 Conclusion

It has been suggested by a number of researchers that the sustained and strong output growth levels observed over the last decade in numerous countries is mainly attributable to the existence of a low and stable inflation environment in these countries. The main purpose of this paper was to examine this question in the case of Canada.

Our methodology consisted of extending Elwood's (1998) unobserved-components framework, where the parameters of output growth switch depending on the sign of own lagged shocks, by allowing for a second threshold effect from the inflation level. Thus, we allowed output growth persistence to evolve according to four possible regimes, depending on the sign of the own lagged shock and on whether inflation was above or below some critical level. We then estimated an ARMA(1,1) specification with the above assumptions using Canadian data on GDP growth and inflation. The estimation methodology made use of the Kalman filter and

maximum likelihood estimates were obtained for the persistence parameters, the mean and variance of the output growth series, as well as for the inflation threshold level.

Results from a standard likelihood ratio test confirmed that positive and negative shocks have significantly different effects on output, similar to the conclusion reached by Beaudry and Koop (1993). On the other hand, testing the hypothesis of no inflation threshold required the simulation of the distribution of the likelihood ratio statistic (following the bootstrap procedure developed in Hansen 1996), which is nonstandard given that the threshold is not identified under the null. The results from this test showed that the inflation threshold effect was also significant.

The above findings were summarized with the calculation of impulse response functions for the different inflation regimes. These showed that shocks to output, whether negative or positive, could be long-lived or temporary, depending on the inflation regime. In particular, we found that a positive shock had a permanent effect on output in a low inflation regime, while a negative shock was highly persistent in a high inflation environment. Similarly, a positive output shock was seen to have a temporary effect when inflation was above its threshold value, whereas the negative shock had more temporary effects in the presence of low inflation.

These results might shed some light on the very different conclusions reached by Campbell and Mankiw (1987) on the one hand, and Clark (1987) on the other, regarding the behavior of US GNP. Using linear ARIMA models, and without distinguishing between positive or negative shocks, the former find that a one percent innovation to current output will change the long-run forecast of this series by more than one percent. However, with a more restricted ARIMA model, Clark finds these shocks to be of a more temporary nature. In fact, if the US series has nonlinearities of the same type as the ones we have explored above, then both

of these studies could be capturing only a part of the behavior of output. We leave this question for future research.

Appendix

This appendix reviews Elwood's (1998) modified Kalman filter as used for the estimation the threshold ARMA(1,1). The state-space representation comprises the observation or measurement equation

$$\Delta y_t = \mu + [1 \ 0] \begin{bmatrix} S_t \\ \varepsilon_t \end{bmatrix} \quad (15)$$

with the state vector $[S_t \ \varepsilon_t]'$ governed by the transition equations

$$\begin{bmatrix} S_t \\ \varepsilon_t \end{bmatrix} = \begin{bmatrix} \phi_1^p & \theta_1^p \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_{t-1} \\ \varepsilon_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \varepsilon_t, \quad \varepsilon_{t-1} \geq 0, \pi_{t-1} \geq d, \quad (16)$$

$$\begin{bmatrix} S_t \\ \varepsilon_t \end{bmatrix} = \begin{bmatrix} \phi_1^n & \theta_1^n \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_{t-1} \\ \varepsilon_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \varepsilon_t, \quad \varepsilon_{t-1} < 0, \pi_{t-1} \geq d, \quad (17)$$

$$\begin{bmatrix} S_t \\ \varepsilon_t \end{bmatrix} = \begin{bmatrix} \phi_2^p & \theta_2^p \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_{t-1} \\ \varepsilon_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \varepsilon_t, \quad \varepsilon_{t-1} \geq 0, \pi_{t-1} < d, \quad (18)$$

$$\begin{bmatrix} S_t \\ \varepsilon_t \end{bmatrix} = \begin{bmatrix} \phi_2^n & \theta_2^n \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_{t-1} \\ \varepsilon_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \varepsilon_t, \quad \varepsilon_{t-1} < 0, \pi_{t-1} < d, \quad (19)$$

where

$$Q = E \left[\begin{bmatrix} \varepsilon_t \\ \varepsilon_t \end{bmatrix} \begin{bmatrix} \varepsilon_t & \varepsilon_t \end{bmatrix} \right] = \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (20)$$

Let $x_t = [S_t \ \varepsilon_t]'$ and denote by $\Phi_1^p, \Phi_1^n, \Phi_2^p, \Phi_2^n$ the 2×2 matrices appearing respectively in the transition equations (16) to (19). The forecast value of x_t on the basis of information available through date $t - 1$, denoted $x_{t|t-1}$, is given by

$$x_{t|t-1} = \Phi_1^p x_{t-1|t-1}, \quad x_{t-1|t-1}^{[2]} \geq 0, \quad \pi_{t-1} \geq d, \quad (21)$$

$$x_{t|t-1} = \Phi_1^n x_{t-1|t-1}, \quad x_{t-1|t-1}^{[2]} < 0, \quad \pi_{t-1} \geq d, \quad (22)$$

$$x_{t|t-1} = \Phi_2^p x_{t-1|t-1}, \quad x_{t-1|t-1}^{[2]} \geq 0, \quad \pi_{t-1} < d, \quad (23)$$

$$x_{t|t-1} = \Phi_2^n x_{t-1|t-1}, \quad x_{t-1|t-1}^{[2]} < 0, \quad \pi_{t-1} < d, \quad (24)$$

where $x_{t-1|t-1}^{[2]}$ corresponds to the second element of $x_{t-1|t-1}$. Let z_t denote the observations on output growth Δy_t and inflation π_t up to date t . Then the distribution of x_t conditional on z_{t-1} is normal with mean $x_{t|t-1}$ and variance $P_{t|t-1}$. The forecast equations for the conditional variance are given by

$$P_{t|t-1} = \Phi_1^p P_{t-1|t-1} \Phi_1^{p(t)} + Q, \quad x_{t-1|t-1}^{[2]} \geq 0, \quad \pi_{t-1} \geq d, \quad (25)$$

$$P_{t|t-1} = \Phi_1^n P_{t-1|t-1} \Phi_1^{n(t)} + Q, \quad x_{t-1|t-1}^{[2]} < 0, \quad \pi_{t-1} \geq d, \quad (26)$$

$$P_{t|t-1} = \Phi_2^p P_{t-1|t-1} \Phi_2^{p(t)} + Q, \quad x_{t-1|t-1}^{[2]} \geq 0, \quad \pi_{t-1} < d, \quad (27)$$

$$P_{t|t-1} = \Phi_2^n P_{t-1|t-1} \Phi_2^{n(t)} + Q, \quad x_{t-1|t-1}^{[2]} < 0, \quad \pi_{t-1} < d, \quad (28)$$

where the superscript (t) denotes the transpose matrix. The filter recursions are such that the estimates of $x_{t-1|t-1}$ and $P_{t-1|t-1}$ are computed before $x_{t|t-1}$ and $P_{t|t-1}$. Therefore, the choice of the appropriate Φ_j^i during each recursion of the filter is unambiguous.

The updating equation for x_t and P_t are given by

$$x_{t|t} = x_{t|t-1} + P_{t|t-1} H e_t / v_t \quad (29)$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1} H H' P_{t|t-1} / v_t \quad (30)$$

with

$$e_t = \Delta y_t - \mu - H' x_{t|t-1} \quad (31)$$

and

$$v_t = H' P_{t|t-1} H \quad (32)$$

where $H' = [1 \ 0]$.

Collect the $n = 11$ model parameters in the vector

$$\theta = (\mu, \sigma^2, \phi_1^p, \phi_1^n, \phi_2^p, \phi_2^n, \theta_1^p, \theta_1^n, \theta_2^p, \theta_2^n, d). \quad (33)$$

Under the assumed model, we have that

$$\Delta y_t | z_{t-1}; \theta \sim N(\mu_t(\theta), \sigma_t^2(\theta)) \quad (34)$$

where

$$\mu_t(\theta) = \mu + H' P_{t|t-1} \quad (35)$$

$$\sigma_t^2(\theta) = v_t \quad (36)$$

Given θ , the above equations can be iterated on to compute the value of the log-likelihood function

$$\sum_{t=1}^T \log f(\Delta y_t | z_{t-1}; \theta) = \quad (37)$$

$$-(Tn/2) \log(2\pi) - (1/2) \sum_{t=1}^T \sigma_t^2(\theta) - (1/2) \sum_{t=1}^T \left(\frac{\Delta y_t - \mu_t(\theta)}{\sigma_t} \right)^2 \quad (38)$$

which, in turn, can be numerically maximized so as to obtain quasi-maximum-likelihood parameter estimates.⁶ See Hamilton (1994) for a general discussion of state-space models and Elwood (1998) for more details on the modified Kalman filter.

⁶Under the maintained assumption $\varepsilon_t \sim i.i.d. N(0, \sigma^2)$, these estimates correspond to the maximum-likelihood parameter estimates.

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