THE EFFECTS OF BACKGROUND RISK ON OPTIMAL PORTFOLIOS

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Abstract

Through this paper we analyze the demand for assets in presence of a dependent background risk (informational asymmetries, political risk, "noisy traders", investment constraints, moral hazard, cross border risk, market risk, non-marketable assets ,....). The desirability of the risky asset is pointed out and also the fact that the demand is monotonously decreasing in the degree of dependence in a specified interval by the use of stochastic variation. Negative dependence is enough to guarantee that investors will take more risky asset even if the expected excess return is null, and this result is reversed in case of positive one. The convexity of the marginal utility is enough to guarantee that the fraction of wealth optimally invested in the risky asset is unambiguously larger (smaller) for wealthier individuals when the dependence is negative (positive).

1.Introduction

The aim of this paper is to develop an analytical framework for investment decision making in globalize international financial markets. To this end we will use some results coming from the area of risk theory. We will also show how this framework can be applied to the particular case of noisy markets.

Our attention will be focused on the dependence between the risky asset and the background risk. Background risk might arise due to any reason like war, earthquakes, moral hazard, informational asymmetries, non-marketable assets (human capital, irreplaceable commodities,....), cross border risk, market risk, political risk, inefficiency .... It is obvious that individuals who invest in countries that have unstable political-economic systems must add a country risk premium when determining their required rates of return. This assertion is true when the background risk, for example the political risk, changes the return directly, otherwise it is false. That is the difference between a multiplicative and an additive risk. On one hand, result of the investment is equal to the investment times a gross return which is the intrinsec gross return of the market plus the political risk. On the other hand, result is given by the intrinsec gross return of the market times the investment, and added to the political risk. In our framework we will consider the last case. Clark and Jokung (1998b) used this framework in order to explain the capital flows between two countries. The background risk could also arise due to the act of “noisy traders” as Krueger (1994) said “Local investors often use the market like a betting on stocks apparently without regard to value. At times, the local activity in emerging markets will be liquidity driven, while at other times, it will be theme driven”. He also said that “The biggest problem (of investing in emerging countries[1]) is probably the lack of reliable information”. Our study will give an analytic explanation of the act of the investment return on the background risk and will also give some implications on the investors' choices.

The past ten years have been marked by many studies on the optimal demand for coverage with random initial wealth or with multiple sources of risk. Ross (1981) showed the weak points of Pratt-Arrow’s risk aversion by using risk whose conditional expectation to random wealth is naught. Kihlstrom, Romer and Williams (1981) studied the same problem with independence between risk and initial wealth. In this case the individual was supposed to have decreasing absolute risk aversion. The first study defined a new measure of risk aversion, more restrictive than Pratt’s, but one that made it possible to solve problems involving many risks, whereas the second one found the same results as Pratt’s (1964). Nachman (1982) extended KRW’s result to non linear payoffs by studying the preservation of “more risk averse” under expectation. Hadar and

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[1] Emerging stock markets have come of age in so far as they have attracted the attention of portfolio investors for the risk diversification and high return they provide. De Santis (1994), Harvey (1995) and other authors document substantial diversification benefits from investing in emerging equity market indices. However little attention is given to emerging markets in the financial literature concerning their behavior. A critical drawback of the studies of De Santis (1994) and Harvey (1995) is their use of market indices that generally ignore investment constraints associated with investments in emerging markets, and therefore ignore the background risk.

Seo (1990) showed the limits of Ross's definition when "more risky" is taken in the sense of Rothchild and Stiglitz (1970-1971) in a portfolio problem.

Some of our results will be similar to those found by Kahane and Kroll (1985) who studied the correlation between a speculative risk and a pure risk by using Sharpe's measure of performance, and also to those of Mayers and Smith (1983) who showed the non-separability of the demand for coverage and the portfolio selection in a mean-variance framework. The latter also demonstrated that the wealthier individual would not necessarily demand more risky asset. Doherty and Schlesinger (1983) pointed out the effect of the correlation between two risks with just one insurable on the demand for coverage and the fact that full insurance will not always be optimal when actuarially fair insurance is available. They used a two-state marginal distribution. Eeckhoudt and Kimball (1992) showed that when the premium is unfair, if the degree of absolute prudence is positive and non increasing with wealth, then for any pair of statistically independent risks the optimal coverage with fixed wealth is less than the optimal coverage with random wealth. Absolute prudence is to marginal utility what absolute risk aversion is to utility and "it measures the propensity to prepare and forearm oneself in face of uncertainty in contrast to absolute risk aversion which is how one dislikes uncertainty and would turn away from uncertainty if one could" as Kimball (1990) said. Eeckhoudt and Kimball (1992) also studied the effect of a background risk on the demand for coverage : in the case of positive dependence the insured reduces his coverage. Clark and Jokung (1998a) applied this result to the capital budgeting decision with political risk and they pointed out the rule of "good times" which means that investors which utility function exhibits decreasing absolute aversion and decreasing absolute prudence prefer to face bad results when the return on the projects is high rather than bad results where the return is lowIn our model, we consider an investor, a risky asset, a risk-less asset, and a political risk or more generally a background risk. We recover Kahane and Kroll's results. The effect of dependence on the demand for risky asset is pointed out. We find that, in fact, it is not necessarily true that the individual invests only in the risk-less asset when the risky asset is fair, where fair means that the expected return of the risky asset is equal to the risk-free rate. In case of separation between low and high returns the demand for risky asset is completely specified. We give the ways to obtain diversified portfolio in both decreasing and increasing stochastic dependence. We analyze the wealth effect by exploring the effect of the background risk on the wealth effect and we showed that decreasing absolute risk aversion combined with decreasing absolute prudence or both risk-aversion and convex marginal utility are enough to guarantee that in the presence of an positive dependent background risk an increase in wealth induces a decrease in the optimal demand for risky asset. And finally we link our framework with the Capital Market Line.

Our paper is organized as follows, section 2 is devoted to the description of the model and the hypotheses. Section 3 deals with the optimal demand for risky assets. Section 4 focuses on the case of high and low returns, and section 5 places emphasis on the effect of an increase in the strength of the dependence. Section 6 presents the wealth effect. In section 7 we apply the analysis to the particularly important case of noisy markets and explicitly show the implication of our results on the Capital Market Line.

2.The model and its assumptions

Consider an investor who owns non risky wealth w_0 which he wants to invest in a risky asset and a risk-less asset. He also faces a second source of risk representing by a background risk. His final wealth is given by :

$$\tilde{W}_f = a(1+\tilde{r}) + (w_0 - a)(1+r_f) + \tilde{e} \quad \text{or} \quad \tilde{W}_f = a(\tilde{r} - r_f) + w_0(1+r_f) + \tilde{e}$$

in which :

a is the amount invested in the risky asset

$\tilde{r}$ is the return of the risky asset

r_f is the risk-free rate

$\tilde{e}$ is the background risk

Given a von Neumann-Morgenstern utility function u with $u' > 0$ and $u'' < 0$, the investor's expected utility following :

$$Eu\left(\tilde{W}_f\right) = Eu\left[a\left(\tilde{r} - r_f\right) + w_0\left(1+r_f\right) + \tilde{e}\right]$$

He is supposed to maximize his expected utility. The first order condition(F.O.C.) for a maximum is :

$$\frac{dEu(\tilde{W}_f)}{da} = E\left[\left(\tilde{r} - r_f\right) u'(\tilde{W}_f)\right] = 0$$

The second order condition (S.O.C.) is always satisfied by the assumed risk-aversion, thus the optimal demand for risky asset is an interior optimum.

The relationships between risky return and background risk can be defined in different manners. See the Appendix for the common definitions.

In order to facilitate the mathematical analysis, we use discrete random variables $\tilde{e}$ and $\tilde{r}$ defined as following :

Hypothesis 1 : $\tilde{r}$ *follows a multinomial law of tail m (m ≥ 2)*

$$P\left(\tilde{r} = r_i\right) = p_i \ \forall i = 1, \ldots m$$

where the r_i (the values taken by $\tilde{r}$)are decreasing in i

Hypothesis 2 : $\tilde{e}$ *follows a multinomial law of tail n (n ≥ 2)*

$$P\left(\tilde{e} = e_j\right) = q_j \ \forall j = 1, \ldots n$$

where the e_j (the values taken by $\tilde{e}$)are increasing in j

Under these hypothesis, we can explain the expected utility with the help of conditional probabilities. Let's take the following notations :

$$p_{\,i,j} = P\left({\tilde{e} = e_{\,j}} \middle/ {\tilde{r} = r_i} \right) \forall i = 1,\ldots,m \;\; \forall j = 1,\ldots,n$$

The expected utility follows :

$$Eu\left(\tilde{W}_f\right) = \sum_{i=1}^{m} p_i \left[\sum_{j=1}^{n} p_{\,i,j}\; u\left(W_i + e_{\,j}\right) \right]$$

where $W_i = a\left(r_i - r_f\right) + w_0\left(1 + r_f\right)$ represents the wealth of the investor when the return of the risky asset is equal to r_i and without the presence of the background risk. And the first order condition becomes :

$$\frac{d\; Eu\left(\tilde{W}_f\right)}{d\; a} = \sum_{i=1}^{m} p_i \left(r_i - r_f\right) \left[\sum_{j=1}^{n} p_{\,i,j}\; u'\left(W_i + e_{\,j}\right) \right] = 0$$

3. Optimality of the risky asset

Now, let's evaluate the first order condition at a = 0 in order to find whether positive demand for risky asset is optimal. First of all, notice that the final wealth in any state (related to $\tilde{r}$) is independent of the return when the individual invests only in the risk-less asset. Recall that $W_i = a\left(r_i - r_f\right) + w_0\left(1 + r_f\right)$ which becomes $w_0\left(1 + r_f\right)$ when a = 0. Hence the evaluation gives :

$$\frac{d\; Eu\left(\tilde{W}_f\right)}{d\; a}\Big|_{\,a=0} = \sum_{i=1}^{m} p_i \left(r_i - r_f\right) \left[\sum_{j=1}^{n} p_{\,i,j}\; \Delta_{\mathrm{j}} \right] \qquad where\, \Delta_{\,j} = u'\left(w_0\,(1 + r_f\,) + e_{\,j}\right)$$

We explore two cases, first the excess return is null ($E\left(\tilde{r} - r_f\right) = 0$) which means that the risky asset is actuarially fair, and second this assumption is relaxed. We need the following lemma:

Lemma 1 : *$(S_i)_{i=1,..,m}$ is monotonically increasing (respectively decreasing) with i when $\tilde{e}$ is stochastically increasing (respectively decreasing) in $\tilde{r}$, with $S_i = \sum_{j=1}^{n} p_{i,j} \Delta_j$ which is the conditional expectation of the marginal utility knowing that the return is r_i when the individual takes only the risk-less asset.*

See the proof at the appendix.

First case: The risky asset is actuarially fair

Let i_0 be the index such that : $\begin{matrix} r_i \geq r_f \text{ when } i \leq i_0 \\ r_i < r_f \text{ when } i > i_0 \end{matrix}$

The fact that $E\left(\tilde{r} - r_f\right) = 0$ induces that some excess returns are negative and the others are positive, thus the index i_0 exists and $i_0 \in [1, m]$. The F.O.C. evaluated at a = 0 becomes :

$$\frac{d\ Eu\left(\tilde{W}_f\right)}{d\ a}\Big|_{a=0} = \sum_{i=1}^{i_0} p_i \left(r_i - r_f\right) S_i - \sum_{i=i_0+1}^{m} p_i \left(r_f - r_i\right) S_i$$

Suppose that $\tilde{e}$ SI $\tilde{r}$ (respectively $\tilde{e}$ SD $\tilde{r}$), then $(S_i)_{i=1,..,m}$ is monotonically increasing (respectively decreasing) with i coming from lemma 1, and we get[2] :

$$\sum_{i=1}^{i_0} p_i (r_i - r_f) S_1 \leq \sum_{i=1}^{i_0} p_i (r_i - r_f) S_i \leq \sum_{i=1}^{i_0} p_i (r_i - r_f) S_{i_0}$$

$$(resp. \sum_{i=1}^{i_0} p_i (r_i - r_f) S_{i_0} \leq \sum_{i=1}^{i_0} p_i (r_i - r_f) S_i \leq \sum_{i=1}^{i_0} p_i (r_i - r_f) S_1)$$

[2] *and*

$$\sum_{i=i_0+1}^{m} p_i (r_f - r_i) S_{i_0+1} \leq \sum_{i=i_0+1}^{m} p_i (r_f - r_i) S_i \leq \sum_{i=i_0+1}^{m} p_i (r_f - r_i) S_i$$

$$(resp. \sum_{i=i_0+1}^{m} p_i (r_f - r_i) S_m \leq \sum_{i=i_0+1}^{m} p_i (r_f - r_i) S_i \leq \sum_{i=i_0+1}^{m} p_i (r_f - r_i) S_{i_0+1})$$

$$\frac{d\ Eu\left(\tilde{W}_f\right)}{d\ a}\Big|_{a=0} \leq \sum_{i=1}^{i_0} p_i\left(r_i - r_f\right)S_{i_0} - \sum_{i=i_0+1}^{m} p_i\left(r_f - r_i\right)S_{i_0+1}$$

$$(resp. \frac{d\ Eu\left(\tilde{W}_f\right)}{d\ a}\Big|_{a=0} \geq \sum_{i=1}^{i_0} p_i\left(r_i - r_f\right)S_1 - \sum_{i=i_0+1}^{m} p_i\left(r_f - r_i\right)S_m\ )$$

And using the following conditions :

$$0 \leq \sum_{i=1}^{i_0} p_i\left(r_i - r_f\right) \leq \sum_{i=i_0+1}^{m} p_i\left(r_f - r_i\right)\ (resp. 0 \geq \sum_{i=1}^{i_0} p_i\left(r_i - r_f\right) \geq \sum_{i=i_0+1}^{m} p_i\left(r_f - r_i\right)\ )$$

$$and \quad 0 \leq S_{i_0} \leq S_{i_0+1} (resp.\ 0 \geq S_1 \geq S_m\ ) \text{ by the use of lemma 1}$$

The right side of the inequality is negative (respectively positive), then :

$\frac{d\ Eu\left(\tilde{W}_f\right)}{d\ a}\Big|_{a=0} \leq\ (\ resp.\ \ \geq)\ 0$ and the optimal demand for risky asset $\hat{a}$ is negative (resp. positive).

<u>Second case</u>: The risky asset is actuarially unfair

The excess return can be expressed as a percentage of the risk free rate, namely :

$E\left(\tilde{r} - r_f\right) = l\ \ r_f\ or\ E\left(\tilde{r}\right) = \left(1 + l\ \right)r_f\ = r_f^*$ which is equivalent to $\sum_{i=1}^{m} p_i\left[r_i - \left(1 + l\ \right)r_f\ \right] = 0$

We can notice that the fair case corresponds to $l\ = 0$.

Let $i_0(\ l\ )$ be the index such that : $\begin{array}{l} r_i \geq (1 + l\ )r_f\ when\ i \leq i_0(\ l\ ) \\ r_i < (1 + l\ )r_f\ when\ i > i_0(\ l\ ) \end{array}$

$i_0(\ l\ )$ depends on l . The first order condition evaluated at a = 0 becomes :

$$\frac{d\ Eu\left(\tilde{W}_f\right)}{d\ a}\Big|_{a=0} = \sum_{i=1}^{m} p_i\left[\left(r_i - \left(1 + l\ \right)r_f\ \right)\right]S_i + l\ \sum_{i=1}^{m} p_i S_i\ \ with\ \sum_{i=1}^{m} p_i\ S_i \geq 0$$

Consider the case of the stochastic increase (respectively decrease) of $\tilde{e}$ in $\tilde{r}$, the first term of the last equation can be rewritten as :

$$\sum_{i=1}^{i_0(l)} p_i\left(r_i - r_f *\right) S_i - \sum_{i=i_0(l)+1}^{m} p_i\left(r_f * - r_i\right) S_i$$

By the same methodology as in the case of actuarially fair risky asset, we get[3] :

$$\sum_{i=1}^{m} p_i\left[\left(r_i - (1+l)\ r_f\right)\right] S_i \le (resp.\ge)0$$

and using the fact that $\sum_{i=1}^{m} p_i S_i$ is non negative , we can sign the slope of the F.O.C. at a = 0. The sign of the demand for risky asset is the following :

	$\tilde{e}$ SI $\tilde{r}$	$\tilde{e}$ SD $\tilde{r}$
$l > 0$	?	positive
$l = 0$	negative	positive
$l < 0$	negative	?

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[3]

$$\sum_{i=1}^{i_0(l)} p_i (r_i - r_f *) S_1 \le \sum_{i=1}^{i_0(l)} p_i (r_i - r_f *) S_i \le \sum_{i=1}^{i_0(l)} p_i (r_i - r_f *) S_{i_0(l)}$$

$$(resp\ .\sum_{i=1}^{i_0(l)} p_i (r_i - r_f *) S_{i_0(l)} \le \sum_{i=1}^{i_0(l)} p_i (r_i - r_f *) S_i \le \sum_{i=1}^{i_0(l)} p_i (r_i - r_f *) S_1\ )$$

$$and$$

$$\sum_{i=i_0(l)+1}^{m} p_i (r_f * - r_i) S_{i_0(l)+1} \le \sum_{i=i_0(l)+1}^{m} p_i (r_f * - r_i) S_i \le \sum_{i=i_0(l)+1}^{m} p_i (r_f * - r_i) S_m$$

$$(resp. \sum_{i=i_0(l)+1}^{m} p_i (r_f * - r_i) S_m \le \sum_{i=i_0(l)+1}^{m} p_i (r_f * - r_i) S_i \le \sum_{i=i_0(l)+1}^{m} p_i (r_f * - r_i) S_{i_0(l)+1}\ )$$

$$where\ \ r_f * = (1+l)\ r_f$$

We group these results in the same proposition :

Proposition 1 : *Under actuarially fair risky asset when the background risk is positively (respectively negatively) dependent on the risky return, a negative (respectively positive) demand for risky asset is optimal. When the risky asset becomes unfair, the result is still the same if the excess return is negative (respectively positive) in case of positive (respectively negative) dependence. Otherwise nothing can be said.*

This proposition summarizes two cases. On one hand, the background is stochastically increasing in the return, which means that small returns are for the most part offset by large gains in the background and the same thing remains for large returns. In the case of actuarially fair risky asset, the expected return for the risky asset is equal to the risk-free rate, usually the agent is indifferent between the two assets and the optimal decision is to invest only in the risk-less asset. But the presence of the background, which is viewed as negative, gives incentives to have a negative demand for risky asset. It is a kind of diversification. Hence the portfolio becomes diversified only when the demand for risky asset is negative. This diversification remains only when the excess return is negative, because a positive one will give incentives to reduce the negative demand, and this demand could become positive due to the marginal utility, the loading factor l, the initial wealth, and the strength of the dependence. A large positive loading factor may induce a positive demand.

On the other hand, when the background is stochastically decreasing in the return, high returns are likely to be accompanied by low values of the background. The background risk acts like a hedge-portfolio, it is a kind of homemade diversification due to the second source of risk. Then the investor has incentives to have a positive demand for risky asset. When the risky asset is positively unfair ($l > 0$), this situation is still valid. In case of negative unfair risky asset, nothing can be said. But with small negative loading factor ($l < 0$), investors could have a positive demand for risky asset when the dependence allows a good homemade diversification.

4. High and Low Returns

Investors are interested with returns which are greater or lower in comparison with the risk-free rate, otherwise they have incentives to invest in the risk-less asset. In this section we study the location of the demand. We need the following assumption :

Hypothesis 3 : r_H *and* r_L *are the values taken by* $\tilde{r}$ *with probabilities* p *and* $1-p$ *respectively such that* $r_L \leq r_f \leq r_H$

We take this assumption in order to avoid any stochastic dominance between the risky asset and the risk-less one. Imagine that $r_L \leq r_H < r_f$ (respectively $r_f < r_L \leq r_H$) then individual will invest only in the risk-less (respectively risky) asset.

In case of actuarially fair risky asset ($p\left(r_H - r_f\right) + (1-p)\left(r_L - r_f\right) = 0$) we are able to locate the optimal demand. Let's denote by $\Delta\tilde{e} = e_n - e_1$ the variability of the background risk, and p_j^H (respectively p_j^L) the conditional probability that the background takes the value e_j knowing that the risky return is r_H (respectively r_L) .

Let $\overline{W}(a) = a\left(r^H - r_f\right) + w_0\left(1 + r_f\right)$ *and* $\underline{W}(a) = a\left(r^L - r_f\right) + w_0\left(1 + r_f\right)$ denoted the level of the final wealth according to the value taken by the risky return. Suppose that $\tilde{e}$ SI $\tilde{r}$ (respectively $\tilde{e}$ SD $\tilde{r}$) and evaluate the F.O.C. at $a^+ = -\dfrac{\Delta\tilde{e}}{r_H - r_L}$ (respectively $a^- = \dfrac{\Delta\tilde{e}}{r_H - r_L}$) :

$$\frac{dEu\left(W_f\right)}{da}\Big|_{a=a^-} = p(r_H - r_f)\left\{\begin{array}{l} \sum_{j=2}^{n} p_j^H\left[u'\left(\overline{W}(a^+) + e_j\right) - u'\left(\overline{W}(a^+) + e_n\right)\right] - \sum_{j=1}^{n-1} p_j^L\left[u'\left(\underline{W}(a^+) + e_j\right) - u'\left(\underline{W}(a^+) + e_n\right)\right] \\ + \left[u'\left(\overline{W}(a^+) + e_n\right) - u'\left(\underline{W}(a^+) + e_1\right)\right] \end{array}\right\}$$

(*respectively*

$$\frac{dEu\left(W_f\right)}{da}\Big|_{a=a^-} = p(r_H - r_f)\left\{\begin{array}{l} \sum_{j=2}^{n} p_j^H\left[u'\left(\overline{W}(a^-) + e_j\right) - u'\left(\overline{W}(a^-) + e_n\right)\right] - \sum_{j=1}^{n-1} p_j^L\left[u'\left(\underline{W}(a^-) + e_j\right) - u'\left(\underline{W}(a^-) + e_n\right)\right] \\ + \left[u'\left(\overline{W}(a^-) + e_1\right) - u'\left(\underline{W}(a^-) + e_n\right)\right] \end{array}\right\}$$

and using the fact that $\overline{W}(a^+) + e_n = \underline{W}(a^+) + e_1$ (respectively $\overline{W}(a^-) + e_1 = \underline{W}(a^-) + e_n$) and that the marginal utility function is decreasing, then we find that the first bracket is positive (respectively negative), the second is negative (respectively positive) and the last one is equal to zero. Hence, the first order condition evaluated at a^+ (respectively a^-) is positive (respectively negative) and the optimal demand for risky asset $\hat{a}$ fulfilled the following condition : $a^+ \leq \hat{a} \leq 0$ (*respectively* $0 \leq \hat{a} \leq a^-$)

We have the following proposition :

Proposition 2 : *Under actuarially fair risky asset. When the background risk is stochastically increasing (respectively decreasing) in the risky return, then the optimal demand for risky asset is negative (respectively positive) and located inside the following interval :*

$$\left[-\frac{\Delta\tilde{e}}{r_H - r_L},\ 0\right] \quad (\textit{respectively}\ \left[0,\ \frac{\Delta\tilde{e}}{r_H - r_L}\right])$$

We can notice that the ratio $\frac{\Delta\tilde{e}}{r_H - r_L} = \frac{e_n - e_1}{r_H - r_L} = \frac{\Delta\tilde{e}}{\Delta\,\tilde{r}}$ is the percentage of additional background risk per asset's risk, we get two cases. First, the background risk is stochastically increasing in the return and the investor uses the risky asset in order to cover the background, then his demand is negative. Second, in the opposite case, the background is a natural coverage for the risky return. And the demand is positive. In both the two cases the absolute value of the demand is lower than this ratio.

5. Increasing in the strength of the dependence

The question which arises here is : "Is the demand for risky asset in presence of $\tilde{e}^{\,2}$ greater (respectively less) than in presence of $\tilde{e}^{\,1}$? " , where $\tilde{e}^{\,1}$ and $\tilde{e}^{\,2}$ are two different background risks taking the same values, $e_{\,1}, e_{\,2}, \ldots\ldots, e_{\,n}$ with the probabilities $q_{\,1}, q_{\,2}, \ldots\ldots, q_{\,n}$ respectively. For sake of clarity let's assume that $\tilde{e}^{\,2}$ is more stochastically increasing (respectively decreasing) in $\tilde{r}$ than $\tilde{e}^{\,1}$, recall that the return of the risky asset follows a binomial law, namely $\tilde{r}$ takes the values r_H and r_L with probabilities p and $1-p$ respectively, and that these returns are such that $r_L \leq .\, r_f \leq r_H$. Let denoted by $p_{\,j}^{\,H(\ell)}$ and $p_{\,j}^{\,L(\ell)}$ the conditional probabilities that $\tilde{e}^{\,\ell}$ takes the values $e_{\,j}$ in case of high return and low return respectively and where $\ell \in [1,2]$ .We know that :

- $\tilde{e}^{\,1}$ is stochastically increasing (respectively decreasing) in $\tilde{r}$
- $\tilde{e}^{\,2}$ is stochastically increasing (respectively decreasing) in $\tilde{r}$
- $\tilde{e}^{\,2}$ is more stochastically increasing (respectively decreasing) in $\tilde{r}$ than $\tilde{e}^{\,1}$

Let $\hat{a}^1$ and $\hat{a}^2$ be the optimal demand for risky asset when the background is $\tilde{e}^{\,1}$ and $\tilde{e}^{\,2}$ respectively[4].

In order to find if $\hat{a}^1$ is greater or lower than $\hat{a}^2$, we are going to evaluate the first order condition which gives $\hat{a}^2$ with $\hat{a}^1$, we get :

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[4] $\hat{a}^1$ and $\hat{a}^2$ are the respective solutions to the following equations :

$$p(r_H - r_f)\sum_{j=1}^{n} p_{\,j}^{\,H^{(1)}} u'\left(\overline{W}(\hat{a}^1) + e_{\,j}\right) + (1-p)(r_L - r_f)\sum_{j=1}^{n} p_{\,j}^{\,L^{(1)}} u'\left(\underline{W}(\hat{a}^1) + e_{\,j}\right) = 0$$

and

$$p(r_H - r_f)\sum_{j=1}^{n} p_{\,j}^{\,H^{(2)}} u'\left(\overline{W}(\hat{a}^2) + e_{\,j}\right) + (1-p)(r_L - r_f)\sum_{j=1}^{n} p_{\,j}^{\,L^{(2)}} u'\left(\underline{W}(\hat{a}^2) + e_{\,j}\right) = 0$$

$$\frac{dEu(W_f)}{da}\bigg|_{a=\hat{a}^1} = p(r_H - r_f)\left\{\sum_{j=1}^{n} p_j^{H(2)} u'\left(\overline{W}(\hat{a}^1) + e_j\right) - \frac{(1-p)(r_f - r_L)}{p(r_H - r_f)} \sum_{j=1}^{n} p_j^{L(2)} u'\left(\underline{W}(\hat{a}^1) + e_j\right)\right\}$$

And by the use of the first order condition which gives $\hat{a}^1$, we can determine $\frac{(1-p)(r_f - r_L)}{p(r_H - r_f)}$. The sign of the last expression is the same as the following :

$$\left\{\sum_{=} {}^{(\ )}u'\left(\overline{W}(\hat{\ }\qquad\right) - \frac{\sum_{=} {}^{(\ )}u'\left(\overline{W}(\quad)\qquad\right)}{\sum {}^{(\ )}u'\left(\underline{W}(\quad)+\qquad\right)} \sum_{=} {}^{(\ )}u\left(\underline{W}\ \hat{}\ )+\qquad\right)\right\}$$

$$\sum_{j=1}^{n} p_j^{H(2)} u'\left(\overline{W}(\hat{a}^1) + e_j\right) \sum_{j=1}^{n} p_j^{L(1)} u'\left(\underline{W}(\hat{a}^1) + e_j\right) - \sum_{j=1}^{n} p_j^{H(1)} u'\left(\overline{W}(\hat{a}^1) + e_j\right) \sum_{j=1}^{n} p_j^{L(2)} u'\left(\underline{W}(\hat{a}^1) + e_j\right)$$

We need the following lemma :

Lemma 2 : *If* $\tilde{e}^{\,2}$ *is more stochastically increasing in* $\tilde{r}$ *than* $\tilde{e}^{\,1}$*, then*

$$0 \le \sum_{j=1}^{n} p_j^{H(2)} u'\left(\overline{W}(a) + e_j\right) \le \sum_{j=1}^{n} p_j^{H(1)} u'\left(\overline{W}(a) + e_j\right) \ \forall\, \mathrm{a} \in \Re$$

$$0 \le \sum_{j=1}^{n} p_j^{L(1)} u'\left(\underline{W}(a) + e_j\right) \le \sum_{j=1}^{n} p_j^{L(2)} u'\left(\underline{W}(a) + e_j\right) \ \forall\, a \in \Re$$

The sense of the inequalities are reversed in case of stochastic decrease.

See the proof at the appendix.

Applying this lemma yields :

$$\underset{(\text{resp.} \ge)}{0 \le} \sum_{j=1}^{n} p_j^{H(2)} u'\left(\overline{W}(\hat{a}^1) + e_j\right) \underset{(\text{resp.} \ge)}{\le} \sum_{j=1}^{n} p_j^{H(1)} u'\left(\overline{W}(\hat{a}^1) + e_j\right)$$

$$0 \leq \sum_{j=1}^{n} p_j^{L(1)} u'\left(\underline{W}(\hat{a}^1) + e_j\right) \leq \sum_{j=1}^{n} p_j^{L(2)} u'\left(\underline{W}(\hat{a}^1) + e_j\right)$$

(resp. ≥) (resp. ≥)

Then the evaluation $\hat{a}^1$ of the first condition which gives $\hat{a}^2$ has a negative (respectively positive) sign. This condition means that $\hat{a}^1$ is greater (respectively lower) than $\hat{a}^2$, because the first order condition which gives $\hat{a}^2$ vanishes at $\hat{a}^2$. We can enunciate the following proposition :

Proposition 3 : *An increase in the positive (respectively negative) dependence reduces (respectively increases) the demand for risky asset in absolute value.*

If we consider the case of actuarially fair risky asset, we know the interval in which the demand is located. Now, we are going to find that the boundaries of this interval correspond to the case of perfect dependence. The positive (respectively negative) perfect dependence means that when the return takes the high (respectively low) value, at the same times the background takes its lowest (greatest) one and vice versa.

We need the following lemma :

Lemma 3 : *In case of actuarially fair risky asset, the optimal demand for risky asset is* $a^+ \ (-\dfrac{e_n - e_1}{r_H - r_L})$ *(respectively* $a^- \ (\dfrac{e_n - e_1}{r_H - r_L})$*.) when the background is perfectly stochastically increasing (respectively decreasing) in the return.*

See the proof at the appendix.

The use of this lemma and the last proposition yields :

Proposition 4 : *In case of actuarially fair risky asset (* $E\left(\tilde{r} - r_f\right) = 0$ *), the demand for asset is an increasing function of the dependence when it goes from perfect positive dependence to perfect negative dependence. The demand goes from* a^+ *to* a^-*.*

See the proof at the appendix.

6. Wealth effect

Totally differentiating the first order condition with respect to w_0 , and a , one obtains:

$$\frac{d\,a}{d\,w_0} = -\frac{\sum_{i=1}^{m} p_i\left(r_i - r_f\right)\left[\sum_{j=1}^{n} p_{\,i,j}\; u''\!\left(W_i + e_{\,j}\right)\right]}{\sum_{i=1}^{m} p_i\left(r_i - r_f\right)^2\left[\sum_{j=1}^{n} p_{\,i,j}\; u''\!\left(W_i + e_{\,j}\right)\right]}$$

and the sign of $\frac{da}{dw_0}$ is given by the numerator. Let's us consider three cases with the return of the risky asset given by the binomial law ($\tilde{r}$ takes the values r_H and r_L with probabilities p and $1-p$ respectively).

We need the following lemma:

Lemma 4: *$T_H(W) = \sum_{j=1}^{n} p_{\,j}^{\,H} d_{\,j}$ is greater (respectively smaller) than $T_L(W) = \sum_{j=1}^{n} p_{\,j}^{\,L} d_{\,j}$ when $\tilde{e}$ is stochastically decreasing (respectively increasing) in $\tilde{r}$, with $d_{\,j} = u''(W + e_{\,j})$*

See the proof at the appendix.

<u>First case</u>: The excess return is equal to zero ($l = 0$)

$E\left[(\tilde{r} - r_f)u''(\tilde{W}_f)\right]$ becomes:

$$p(r_H - r_f)\left\{\sum_{j=1}^{n} p_{\,j}^{\,H(2)} u''\!\left(\overline{W} + e_{\,j}\right) - \sum_{j=1}^{n} p_{\,j}^{\,L(2)} u''\!\left(\underline{W} + e_{\,j}\right)\right\} = p(r_H - r_f)\left(T_H(\overline{W}) - T_L(\underline{W})\right)$$

If the background risk is stochastically increasing in the risky return, then the demand for risky asset is negative and $\overline{W} < \underline{W}$. Therefore $T_L(\overline{W}) < T_L(\underline{W})$ using the fact that $T_L(W)$ and $T_H(W)$ are both increasing with wealth. But $T_H(\overline{W}) < T_L(\underline{W})$, hence $T_H(\overline{W}) < T_L(\underline{W})$ and $\frac{da}{dw_0} < 0$.

If the background risk is stochastically decreasing in the risky return, then the demand for risky asset is positive and $\overline{W} > \underline{W}$. Therefore $T_H(\overline{W}) < T_H(\underline{W})$. But $T_H(\overline{W}) > T_L(\underline{W})$, hence $T_H(\overline{W}) < T_L(\underline{W})$ and $\frac{da}{dw_0} > 0$.

<u>2nd case</u>: The excess return is positive ($l > 0$)

$$E\left[(\tilde{r} - r_f)u''(\tilde{W}_f)\right] = p(r_H - r_f)\left(T_H(\overline{W}) - T_L(\underline{W})\right) - l\ r_f T_L(\underline{W})$$

If the background risk is stochastically decreasing in the risky return, then $\frac{da}{dw_0} > 0$ using the fact that $T_L(\underline{W}) < 0$. When the background risk is stochastically increasing in the risky return, nothing can't be said about the sign of $\frac{da}{dw_0}$.

<u>3rd case</u>: The excess return is negative ($l < 0$)

If the background risk is stochastically increasing in the risky return, then $\frac{da}{dw_0} < 0$ using the fact that $T_L(\underline{W}) < 0$. Otherwise nothing can be said.

We get the following proposition:

Proposition 5: *When the utility function is such that* $u' > 0, u'' < 0, u''' > 0$*, in case of actuarially fair risky asset, the demand for risky asset is an inferior good (respectively normal good) when the background risk is positively (respectively negatively) dependent on the risky asset. When the risky asset becomes unfair, the result still the same if the excess return is negative (respectively positive) in case of positive (respectively negative) dependence.*

We point out the fact that in case of fair risky asset, the fraction of wealth optimally invested in the risky asset is unambiguously larger for wealthier individuals when the background risk and asset return risk are negatively dependent, therefore the investor's willingness to hold risky asset in her portfolio increases. We shown that positive relation between background risk and asset return in the sense of the stochastic variation reduces the investor's willingness to hold the risky asset for any utility function that exhibits decreasing absolute risk-aversion and decreasing absolute prudence (which imply $u' > 0, u'' < 0, u''' > 0$ *).*

7. Implications on the CML

Consider a specific market, and denote by $\tilde{r}_m$ the return of his market portfolio and by r_f the risk-free rate. By the use of the theorem of the two funds, an investor will take a portfolio on the market line, this portfolio is given by $a\ \tilde{r}_m + (1-a)\ r_f$, where a is the fraction of wealth invested in the risky asset. We have assumed that the investors buy securities and that they bear no risk apart from securities' risk. This is unrealistic since every investor has to bear some non-tradable risks or to support exogenous risks. When the investor takes into account the background risk $\tilde{e}$ related to this particular country, his portfolio must become $a\ (\tilde{e})\ \tilde{r}_m + (1-a(\tilde{e})\ )\ r_f$.

Recall that the condition which gives the demand for risky asset is the following :

$$E[\tilde{r}_m - r_f] = Cov\left(\tilde{r}_m - r_f, -\frac{u'(\tilde{W}_f)}{Eu'(\tilde{W}_f)}\right) \text{or } E(\tilde{r}_m) - r_f = Cov\left(\tilde{r}_m, -\frac{u'(\tilde{W}_f)}{Eu'(\tilde{W}_f)}\right)$$

with $\tilde{W}_f = a(\tilde{r}_m - r_f) + w_0(1+r_f)$ if the investor does not take into account the background risk else $\tilde{W}_f = a(\tilde{r}_m - r_f) + w_0(1+r_f) + \tilde{e}$. The implication of the former sections in portfolio choice in emerging market is that an increase (a decrease) in the background risk lowers (rises) the optimal amount of money invested in the market portfolio.

Suppose that $a(0)$ is the optimal fraction of wealth invested in the market portfolio if the background risk is non-existent, $a(0)$ is given by $E\lfloor (\tilde{r}_m - r_f)u'\lfloor a(0)w_0(\tilde{r}_m - r_f) + w_0(1+r_f)\rfloor \rfloor = 0$.

If the background exist, but a has not been changed ($a = a(0)$), then $E\lfloor (\tilde{r}_m - r_f)u'\lfloor a(0)w_0(\tilde{r}_m - r_f) + w_0(1+r_f) + \tilde{e}\rfloor \rfloor$ will not be equal to zero. Hence a needs to be lowered or risen to obtain the optimal value, namely $a(\tilde{e})$, this indicates that the investor willingness to buy marketable risk is modified by the presence of the background risk.

That induces a right (left) shift in the Capital Market Line (CML). Investors not only have to add a risk premium in order to eliminate the background risk, but also have to notice that this risk is not proportional according to the movement of the CML.

8. Conclusion

Our study contributes to highlight the interdependence between the background risk and the portfolio selection. When the background is stochastically decreasing in the risky asset, it acts like a hedge-wealth by partially diversifying the portfolio. The use of stochastic variation allows us to focus on this role of natural hedge through the strength of the dependence. This strength acts on the demand for risky asset by increasing it under positive dependence and by decreasing it in the opposite case : the demand for risky asset is monotonically continuously increasing when the degree of dependence goes from perfect positive to perfect negative. With high and low returns, the demand for risky asset is located in a particular interval and its absolute value is still lower than the percentage of background risk per risky asset. We shown that wealthier individuals demand less risky asset when the dependence is negative due to the convexity of the marginal utility function. Dealing with noisy markets, we pointed out that the demand is shifted in order to diversify the international portfolio.

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Appendix

Relationships between risky return and background risk can be defined in different manners. Let $\Re$ be the set of real numbers and let $\tilde{r}$ and $\tilde{e}$ be two random variables on the probability space $(\Re, B(\Re), P(.))$, we have the following definitions for different sorts of dependence :

Definition 1 : *$\tilde{e}$ is stochastically increasing in $\tilde{r}$ (denoted by $\tilde{e}$ SI $\tilde{r}$) if and only if $P\left(\tilde{e} \le e \middle/ \tilde{r} \le r\right)$ is decreasing in r.*

For a given e the probability that the background $\tilde{e}$ is lower than e is larger the return r is lower

***Definition 2* :** *$\tilde{e}$ is stochastically decreasing in $\tilde{r}$ (denoted by $\tilde{e}$ SD $\tilde{r}$) if and only if $P\left(\tilde{e} \le e \middle/ \tilde{r} \le r\right)$ is increasing in r.*

Definition 3 *:Given three random variables $\tilde{r}$, $\tilde{e}$ and $\tilde{e}'$, $\tilde{e}'$ is said to be more stochastically increasing in $\tilde{r}$ than $\tilde{e}$ if and only if :*

i) *$\tilde{e}$ is stochastically increasing in $\tilde{r}$*

ii) $\tilde{e}'$ is stochastically increasing in $\tilde{r}$

iii) $P\left(\tilde{e}' \le e \middle/ \tilde{r} = r\right) \le P\left(\tilde{e} \le e \middle/ \tilde{r} = r\right) \quad \forall r, e$

This definition will allow us to compare the effect of two different background risks on the same return.

***Definition 4* :** *Given three random variables $\tilde{r}$, $\tilde{e}$ and $\tilde{e}'$, $\tilde{e}'$ is said to be more stochastically decreasing in $\tilde{r}$ than $\tilde{e}$ if and only if*

i) $\tilde{e}$ is stochastically decreasing in $\tilde{r}$

ii) $\tilde{e}'$ is stochastically decreasing in $\tilde{r}$

iii) $P\left(\tilde{e}' \le e \middle/ \tilde{r} = r\right) \ge P\left(\tilde{e} \le e \middle/ \tilde{r} = r\right) \quad \forall r, e$.

***Definition 5* :** *Given two random variables $\tilde{r}$ and $\tilde{e}$, $\tilde{r}$ and $\tilde{e}$ are said to be positive dependent in the sense of the likelihood ratio (or positively likelihood ratio dependent, denoted by PLRD) if and only if*

$$P(\tilde{e} \le e,\ \tilde{r} \le r')P(\tilde{e} \le e',\ \tilde{r} \le r) \le P(\tilde{e} \le e',\ \tilde{r} \le r')\ P(\tilde{e} \le e, \tilde{r} \le r)\ \forall e' \le e\ , r' \le r$$

$$or \quad \frac{P(\tilde{e} \leq e \ , \tilde{r} \leq r')}{P(\tilde{e} \leq e', \tilde{r} \leq r')} \leq \frac{P(\tilde{e} \leq e, \tilde{r} \leq r)}{P(\tilde{e} \leq e', \tilde{r} \leq r)} \quad \forall e' \leq e, \ r' \leq r.$$

The negative dependence is obtained by reversing the sense of the inequality. It is denoted by NLRD. We close these different types of dependence by one obtained by the means of the stochastic dominance criteria

Definition 6 : *Given two random variables $\tilde{r}$ and $\tilde{e}$, $\tilde{e}$ is positively (respectively negatively) dependent on $\tilde{r}$ in the sense of the first, the second and the third stochastic dominance criteria respectively if and only if for any pair (r, r') such that $r' \leq r$, $\tilde{e}(r')$ is stochastically dominated (respectively dominates) $\tilde{e}(r)$ by the first, the second and the third stochastic dominance criteria respectively.*

Where $\tilde{e}(r)$ denoted the random variable obtained by conditioning $\tilde{e}$ by $\tilde{r} = r$, the random variable $\tilde{e}(r)$ takes the same values as $\tilde{e}$, but with different probabilities. If the variables are discrete, the law of $\tilde{e}(r)$ is given by $P(\tilde{e}(r) = e_i) = P\left(\tilde{e} = e_i \middle/ \tilde{r} = r\right)$ where the e_i are the values taken by $\tilde{e}$. In our study we use definition 1 through definition 4 (the results obtained by the use of definition 1 through definition 4 are still valid when dependence is defined by the likelihood ratio) because of the following property :

Property 1 : *Given two random variables $\tilde{r}$ and $\tilde{e}$. If $\tilde{r}$ and $\tilde{e}$ are positively (respectively negatively) dependent in the sense of the likelihood ratio, then $\tilde{e}$ is stochastically increasing (respectively decreasing) in $\tilde{r}$*

Proof. Suppose that $\tilde{r}$ and $\tilde{e}$ are positive dependent in the sense of the likelihood ratio (or positively likelihood ratio dependent, denoted by PLRD) :

$$\frac{P(\tilde{e} \leq e \ , \tilde{r} \leq r')}{P(\tilde{e} \leq e', \tilde{r} \leq r')} \leq \frac{P(\tilde{e} \leq e, \tilde{r} \leq r)}{P(\tilde{e} \leq e', \tilde{r} \leq r)} \ \forall e' \leq e, r' \leq r.$$

For sake of clarity, let $\mathrm{r} \in \{ r_{H,} \ r_L \}$. The former condition becomes :

$$\frac{\sum_{k=1}^{i} p_k^L}{\sum_{k=1}^{i} p_k^L} \le (resp. \ge) \frac{\sum_{k=1}^{j} p_k^H}{\sum_{k=1}^{j} p_k^H} \ \forall\, i, j \in [1, n] \ such\ that \ \ j \le i$$

but when i is equal to n, we have : $\sum_{k=1}^{n} p_k^L = \sum_{k=1}^{n} p_k^L$.Hence with i = n the condition becomes

$$\sum_{k=1}^{j} p_k^L \ge (resp. \le) \sum_{k=1}^{j} p_k^H \ \forall\ j = 1, n$$

which means that $P(\tilde{e} \le e \big/ \tilde{r} = r_{H,}) \ \le \ (resp. \ \ge) \ P(\tilde{e} \le e \big/ \tilde{r} = r_L) \ \ \forall\ e$

And $\tilde{e}$ is stochastically increasing (respectively decreasing) in $\tilde{r}$. The controversy of this property is false, because the dependence in the sense of the likelihood ratio is symmetrical and the stochastic variation is not symmetrical.

Lemma 1**:** *$(S_i)_{i=1,..,m}$ is monotonically increasing (respectively decreasing) with i when $\tilde{e}$ is stochastically increasing (respectively decreasing) in $\tilde{r}$, with $S_i = \sum_{j=1}^{n} p_{i,j} \Delta_j$ which is the conditional expectation of the marginal utility knowing that the return is r_i when the individual takes only the risk-less asset.($\Delta_j = u'\left(w_0(1 + r_f) + e_j\right)$)*

Proof.

$$S_{i+1} - S_i = \left[p_{i+1,1} - p_{i,1}\right][\ \Delta_1 - \Delta_2\ ] + \left[(p_{i+1,1} + p_{i+1,2}) - (p_{i,1} + p_{i,2})\right][\ \Delta_2 - \Delta_3\ ]$$

$$+ \left[(p_{i+1,1} + p_{i+1,2} + p_{i+1,3}) - (p_{i,1} + p_{i,2} + p_{i,3})\right][\ \Delta_3 - \Delta_4\ ] + \dots\dots\dots\dots$$

$$+ \left[(p_{i+1,1} + p_{i+1,2} + \dots\dots + p_{i+1,n-1}) - (p_{i,1} + p_{i,2} + \dots\dots + p_{i,n-1})\right][\ \Delta_{n-1} - \Delta_n\ ]$$

$$= \sum_{k=1}^{n-1} \left\{ \left[\ (\sum_{j=1}^{k} p_{i+1,j}) - (\sum_{j=1}^{k} p_{i,j})\ \right] [\ \Delta_k - \Delta_{k+1}\ ] \right\}$$

But the fact that $\tilde{e}$ is stochastically increasing (resp. decreasing) in $\tilde{r}$ tells us that :

$$\sum_{j=1}^{k} p_{1,j} \leq \sum_{j=1}^{k} p_{2,\mathrm{j}} \leq \sum_{j=1}^{k} p_{1,3} \leq \ldots\ldots\ldots\ldots \leq \sum_{j=1}^{k} p_{\mathrm{m,j}} \quad \forall \text{ k } = 1,\ldots, \text{n-1}$$

(resp.≥) (resp.≥) (resp.≥) (resp.≥)

Then :

$$\left[p_{i+1,1} - p_{i,1}\right] \geq 0$$

(resp. ≤ 0)

$$\left[(p_{i+1,1} + p_{i+1,2}) - (p_{i,1} + p_{i,2})\right] \geq 0$$

(resp. ≤ 0)

$$\left[(p_{i+1,1} + p_{i+1,2} + p_{i+1,3}) - (p_{i,1} + p_{i,2} + p_{i,3})\right] \geq 0$$

……….

$$\left[(p_{i+1,1} + p_{i+1,2} + \ldots\ldots + p_{i+1,n-1}) - (p_{i,1} + p_{i,2} + \ldots\ldots + p_{i,n-1})\right] \geq 0$$

(resp. ≤ 0)

or equivalently :

$$\left(\sum_{j=1}^{k} p_{i+1,j} \right) - \left(\sum_{j=1}^{k} p_{i,j} \right) \geq 0 \quad \forall\, k = 1,\ldots,n$$

(resp. ≤ 0)

But $\left(\Delta_j\right)_{j=1,n}$ is decreasing with j due to the fact that the marginal utility is decreasing with wealth and $\left(e_j\right)_{j=1,n}$ is increasing with j, then we :

$$\left[\left(\sum_{j=1}^{k} p_{i+1,j} \right) - \left(\sum_{j=1}^{k} p_{i,j} \right) \right] \left(\Delta_k - \Delta_{k+1}\right) \geq (resp.\ \leq) \quad 0 \quad \forall\ k = 1,\ldots,n$$

And the result follows.

Lemma 2 : *If $\tilde{e}^{2}$ is more stochastically increasing in $\tilde{r}$ than $\tilde{e}^{1}$, then*

$$0 \le \sum_{j=1}^{n} p_j^{H(2)} u'\left(\overline{W}(a)+e_j\right) \le \sum_{j=1}^{n} p_j^{H(1)} u'\left(\overline{W}(a)+e_j\right) \ \forall \text{ a} \in \Re$$

$$0 \le \sum_{j=1}^{n} p_j^{L(1)} u'\left(\underline{W}(a)+e_j\right) \le \sum_{j=1}^{n} p_j^{L(2)} u'\left(\underline{W}(a)+e_j\right) \ \forall a \in \Re$$

The sense of the inequalities are reversed in case of stochastic decrease.

Proof. Let $\overline{W}^{\ell}(a) = a(r_{\ell} - r_f) + w_0(1+r_f) \textit{ where } \ell \in \{H, L\}$

$$\sum_{j=1}^{n} p_j^{H(2)} u'\left(\overline{W}^{\ell}(a)+e_j\right) - \sum_{j=1}^{n} p_j^{H(1)} u'\left(\overline{W}^{\ell}(a)+e_j\right) = \sum_{j=1}^{n} (p_j^{H(2)} - p_j^{H(1)})\left[u'\left(\overline{W}^{\ell}(a)+e_j\right)\right]$$

$$= \sum_{j=1}^{n} (p_j^{H(2)} - p_j^{H(1)})\left[u'\left(\overline{W}^{\ell}(a)+e_j\right) - u'\left(\overline{W}^{\ell}(a)+e_n\right)\right]$$

$$= \sum_{\text{k}=1}^{\text{n-1}} \left\{ \left[\left(\sum_{j=1}^{k} p_j^{H(2)}\right) - \left(\sum_{j=1}^{k} p_j^{H(1)}\right) \right] \left[\text{Z}_k - \text{Z}_{k+1} \right] \right\}$$

where $\text{Z}_\text{k} = u'(\overline{W}^{\ell}(a)+e_k) - u'(\overline{W}^{\ell}(a)+e_n)$

But the fact that $\tilde{e}^{\,2}$ is more stochastically increasing (resp. decreasing) in $\tilde{r}$ than $\tilde{e}^{\,1}$ induces :

$$P(\tilde{e}^{\,2} \le e \,/\, \tilde{r} = r) \ \le \ (resp. \ \ge) \ P(\tilde{e}^{\,1} \le e \,/\, \tilde{r} = r) \ \ \forall \ e, r$$

or equivalently:

$$P(\tilde{e}^{\,2} \le e_j \,/\, \tilde{r} = r) \ \le \ (resp. \ \ge) \ P(\tilde{e}^{\,1} \le e_j \,/\, \tilde{r} = r) \ \ \forall\, j \in [1, n] \text{ for } \text{r} \in \{ r_{H,} \ r_L \}$$

And with the conditional probabilities :

$$\left(\sum_{j=1}^{k} p_j^{H(2)}\right) \ \le \ (\text{resp.}\ge) \ \left(\sum_{j=1}^{k} p_j^{H(1)}\right) \ \forall \text{ k } = 1, \ldots, \text{ n-1}$$

$$\left(\sum_{j=1}^{k} p_j^{L(2)}\right) \ \le \ (\text{resp. } \ge) \ \left(\sum_{j=1}^{k} p_j^{L(1)}\right) \ \forall \text{ k } = 1, \ldots, \text{ n-1}$$

Then

$$\Big(\sum_{j=1}^{k} p_j^{H(2)} \Big) - \Big(\sum_{j=1}^{k} p_j^{H(1)} \Big) \leq (resp. \geq)\ 0 \quad \forall\, k \in [\,1, n-1\,]$$

But the Z_k are positive and decreasing (respectively increasing) due to the fact that the marginal utility is a decreasing function of wealth; thus :

$$\left[\Big(\sum_{j=1}^{k} p_j^{H(2)} \Big) - \Big(\sum_{j=1}^{k} p_j^{H(1)} \Big) \right] \left(\mathrm{Z}_k - \mathrm{Z}_{k+1} \right) \leq (resp. \geq)\ 0 \quad \forall\, k \in [\,1, n-1\,]$$

And finally :

$$\sum_{k=1}^{n-1} \left\{ \left[\Big(\sum_{j=1}^{k} p_j^{H(2)} \Big) - \Big(\sum_{j=1}^{k} p_j^{H(1)} \Big) \right] \left(\mathrm{Z}_k - \mathrm{Z}_{k+1} \right) \right\} \leq (resp. \geq)\ 0$$

Lemma 3 : *In case of actuarially fair risky asset, the optimal demand for risky asset is* $a^{+} \left(-\frac{e_n - e_1}{r_H - r_f} \right)$ *(respectively* $a^{-} \left(\frac{e_n - e_1}{r_H - r_f} \right)$ *.) when the background is perfectly stochastically increasing (respectively decreasing) in the return.*

Proof. Positive (respectively negative) perfect dependence means that when the return takes the high (respectively low) value, at the same times the background takes its lowest (greatest) one and vice versa. Suppose that the dependence is perfect and positive (respectively negative), the amount invested in the risky asset is given by the following equation :

$$p\left(r_H - r_f\right) u'\left[a\left(r_H - r_f\right) + e_1\right] + (1-p)\left(r_L - r_f\right) u'\left[a\left(r_L - r_f\right) + en\right] = 0$$
$$\left(resp. \, p\left(r_H - r_f\right) u'\left[a\left(r_H - r_f\right) + e_n\right] + (1-p)\left(r_L - r_f\right) u'\left[a\left(r_L - r_f\right) + e_1\right] = 0 \ \right)$$

but $p\left(r_H - r_f\right) = -(1-p)\left(r_L - r_f\right)$

then $u'\left[a\left(r_H - r_f\right) + e_1\right] = u'\left[a\left(r_L - r_f\right) + e_n\right] (\mathrm{resp.} u'\left[a\left(r_H - r_f\right) + e_n\right] = u'\left[a\left(r_L - r_f\right) + e_1\right] \ )$

Hence $a\left(r_H - r_f\right) + e = a\left(r_L - r_f\right) + e_n \ (resp. \, a\left(r_H - r_f\right) + e_n = a\left(r_L - r_f\right) + e_1 \ )$

$$\text{and } a = -\frac{e_n - e_1}{r_H - r_f} = a^- \,(resp. a = \frac{e_n - e_1}{r_H - r_f} = a^+)$$

Lemma 4**:** $T_H(W) = \sum_{j=1}^{n} p_j^{\ H} d_j$ *is greater (respectively smaller) than* $T_L(W) = \sum_{j=1}^{n} p_j^{\ L} d_j$ *when* $\tilde{e}$ *is stochastically decreasing (respectively increasing) in* $\tilde{r}$ *, with* $d_j = u''(W + e_j)$

Proof. $T_H(W) - T_L(W) = \sum_{j=1}^{n} p_j^{\ H} d_j - \sum_{j=1}^{n} p_j^{\ L} d$

$$\begin{aligned} T_H(W)\text{-}T_L(W) &= \left[p_1^H - p_1^L\right]\left[d_1 - d_2\right] + \left[(p_1^H + p_2^H) - (p_1^H + p_2^L)\right]\left[d_2 - d_3\right] \\ &+ \left[(p_1^H + p_2^H + p_3^H) - (p_1^L + p_2^L + p_3^L)\right]\left[d_3 - d_4\right] + \dots\dots\dots \\ &+ \left[(p_1^H + p_2^H + \dots\dots + p_{n-1}^H) - (p_1^L + p_2^L + \dots\dots + p_{n-1}^L)\right]\left[d_{n-1} - d_n\right] \end{aligned}$$

$$= \sum_{k=1}^{n-1} \left\{ \left[\left(\sum_{j=1}^{k} p_j^H\right) - \left(\sum_{j=1}^{k} p_j^L\right) \right] \left[d_k - d_{k+1}\right] \right\}$$

But the fact that $\tilde{e}$ is stochastically increasing (respectively decreasing) in $\tilde{r}$ tells us that the term between the brackets is positive (respectively negative). And using the fact that the marginal utility function is convex tells us that $d_k - d_{k+1} = u''(W + e_k) - u''(W + e_{k+1})$ is non positive. Therefore, we get the result.