Financial Innovation, Market Participation and Asset Prices[*]

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Abstract

This paper proposes that an important consequence of financial innovation is to induce endogenous changes in the number and composition of market participants. Under plausible assumptions, these participation changes induce an increase in the relative price of pre-existing assets, and this effect is stronger for assets in the direction of innovation. We introduce the concept of GEI equilibrium with endogenous participation (GEEP). Competitive investors can freely operate in the credit market but must pay a fixed entry cost to invest in risky assets. Asset prices and the participation structure are jointly determined in equilibrium. We show existence and constrained optimality of equilibrium under general conditions, and then specialize to a CARA-normal framework with finitely many risk factors. When new entrants have low risk exposure to the risk spanned by an existing asset, the mean endowment of market participants becomes less correlated with the asset, leading to a decrease in its risk premium and an increase in its relative price. These theoretical results are shown to be consistent with the empirical literature on financial innovation.

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1. Introduction

This paper proposes that an important consequence of financial innovation is to induce endogenous changes in the number and composition of market participants, which lead, under plausible assumptions, to an increase in the relative price of preexisting assets. This result has a remarkably simple intuition in a mean variance setting. When new entrants have low risk exposure to the risk spanned by an existing asset, the mean endowment of market participants becomes less correlated with the asset, leading to a decrease in its risk premium and an increase in its relative price.

Our approach builds on two stylized facts. First, participation in financial markets is costly. Corporate hedging requires the employment of experts able to effectively reduce the firm's risk exposure using existing financial assets. Investors have to sustain learning costs and costs related to the opening and maintenance of accounts with an exchange or a brokerage firm. Statutory and government regulations often create barriers to participation of institutional investors to some financial assets. Second, in an economy with incomplete markets, financial innovation affects the available risk sharing and investment opportunities. The introduction of options and futures allows insurance against the price risk of the underlying assets and commodities. Similarly, mortgages, credit card receivables, airplane and car leasing contracts are sold to investors in the form of asset-backed securities and allow lending institutions to greatly reduce their risk exposure.

We consider a two period finance economy with incomplete markets and endogenous participation. Agents can freely operate in the credit market but have to pay a fixed entry cost in order to invest in one or more risky assets. Financial innovation affects asset prices through increased spanning and the modified composition of market participants. We specify our model to a CARA-normal framework with a factor structure for the agents' risk exposure. The effect of financial innovation on asset prices depends on the cross-sectional distribution of risk across agents. We find that the introduction of new securities tends to increase the relative prices of existing assets when the majority of agents are endowed with low risk exposure. Consistent with the empirical literature on the introduction of new options[1], we show that the increase in relative price is strongest for assets which are most closely correlated to the new asset.

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[1] See Conrad (1989), Detemple and Jorion (1990), Stucki and Wasserfallen (1994). Jochum and Kodres (1998) review the empirical literature on the effects of the introduction of derivatives on the price of the underlying asset.

Section 2 introduces and analyzes the concept of GEI equilibrium with endogenous participation (GEEP) in an arbitrary two period setting. We then specialize to the CARA-normal framework. Section 3 analyzes the entry decision of an investor and derives the equilibrium conditions. Sections 4 introduces a factor model for agents' risk exposure. Section 5 and 6 examine the effect of financial innovation under various factor structures.

1.1. Review of Previous Literature

Restricted participation models, with a fixed financial structure, have been used to explain the equity premium puzzle (Mehra and Prescott, 1985). Mankiw and Zeldes (1991) and Vissing-Jørgensen (1998) study empirical issues whereas Basak and Cuoco (1998) provide a theoretical framework. Endogenous participation has also been used in the asset pricing literature to analyze other issues: volatility of asset prices (Pagano, 1989; Allen and Gale, 1994; Orosel, 1998), market liquidity (Williamson, 1993), futures risk premia (Hirshleifer, 1989). Another strand of the asset pricing literature, pioneered by Allen and Gale (1988), focus on the effect of financial innovation on asset prices, without consideration to participation issues. Detemple and Selden (1991) show, in a general one period economy, that there are interactions between the option and the stock market. They also show that, under some preference and informational assumptions, the introduction of a call option increases the underlying stock price. Huang and Wang (1997) find, in a dynamic model, that the introduction of a collar contract increases the volatility and the price of the underlying stock. Calvet (1997) studies how the evolution of the economy financial structure can determine fluctuations in asset prices. To the best of our knowledge, financial innovation and market participation have been analyzed mainly as separate issues in the literature. We have found two notable exceptions. Allen and Gale (1990) consider a model with endogenous participation to an option exchange. They show that the market structure of the economy need not be efficient when the decision to set-up an option exchange is taken by a monopolist before charging entry fees from investors willing to participate in the new market. Pagano (1993) endogenizes financial innovation in the form new stock issues. In his model investors and firms face entry costs to participate in the stock market. He argues that the existence of multiple equilibria might explain why different stock market sizes are observed across countries and time.

2. GEI Equilibrium with Endogenous Participation

We examine an exchange economy with two periods ($t = 0; 1$) and a single perishable good. The economy is stochastic, and all random variables are defined on a probability space $(\Omega; F; P)$. There exists a set H of agents who consume the good in every date $t = 0; 1$. During his life, each agent $h \in H$ receives an exogenous random endowment $e^h = \left(e_0^h; \tilde{e}^h\right)$; which corresponds for instance to a random labor income. Investors have preferences over consumption streams $c^h = (c_0^h; \tilde{c}^h)$, which are represented by a continuous and strictly concave utility function $U^h(c_0^h; \tilde{c}^h)$.

In this paper, we place no restriction on the size of the set H containing all agents. In particular, the set H can be finite or infinite. To provide a uniform treatment, we endow the space H with a measure μ that satisfies $\mu(H) = 1$: This is equivalent to viewing each element of H as a type, and to view μ as a probability measure on all possible types.

At date $t = 0$, agents can exchange two types of real securities. First, there exists a riskless asset costing π_0 in date $t = 0$ and delivering one unit of the good with certainty at date $t = 1$. Second, there also exist J risky assets ($j = 1; :::J$) with price π_j and random payoff $\tilde{a}_j$: We assume for simplicity that all assets are in zero net supply.[2] Investors can freely operate in the bond market but have to pay a fixed entry cost in order to invest in one or more risky assets. Investors are price-takers both in their entry and portfolio decisions, and there are no constraints on short sales. Denote by π the vector of risky asset prices, by θ^h the vector of risky assets bought (or sold) by investor h, and by $1_{\{\theta^h \neq 0\}}$ the indicator function equal to 1 if $\theta^h \neq 0$ and 0 otherwise. With this notation, the budget set of agent h is defined by

$$B^h(\pi_0; \pi) = \left\{ \left(c_0; \tilde{c}; \theta_0^h; \theta^h\right) : \begin{array}{l} c_0 + \pi_0\theta_0^h + \pi \cdot \theta^h + 1_{\{\theta^h \neq 0\}} \leq e_0^h \\ \tilde{c} \leq \tilde{e}^h + \theta_0^h + \tilde{a} \cdot \theta^h \end{array} \right\} :$$

We note that this budget set can be viewed as the union of two closed sets. Given a participation decision, an agent can calculate the optimal consumption-portfolio choice and therefore an indirect utility level. The agent then chooses the participation decision with the highest indirect utility.

The equilibrium concept is defined as follows.

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[2] This framework is particularly pertinent for securities such as futures and options. Extensions of this model would consider the endogenous supply of securities by firms.

Definition 1. We say that $\left[\pi_0; \pi; \left(\theta_0^h; \theta^h; c_0^h; \tilde{c}^h\right)_{h \in H}\right]$ is a General Equilibrium with Endogenous Participation (GEEP) if

1. $\int_H (c_0^h + 1_{\{\theta^h \neq 0\}}) d\mu(h) = \int_H e_0^h d\mu(h)$; and $\int_H \tilde{c}^h(\omega) d\mu(h) = \int_H \tilde{e}^h(\omega) d\mu(h)$ for all $\omega \in \Omega$.
2. $\int_H \theta_j^h d\mu(h) = 0$ for all $j = 0, \ldots, J$.
3. $\left(c_0^h; \tilde{c}^h; \theta_0^h; \theta^h\right) \in B^h(\pi_0; \pi)$ for all $h \in H$.
4. $\left[(c_0; \tilde{c}; \theta_0; \theta) \in B^h(\pi_0; \pi)\right] \Rightarrow \left[U^h(c_0; \tilde{c}) \leq U^h\left(c_0^h; \tilde{c}^h\right)\right]$ for all $h \in H$.

In the absence of entry costs ($\;= 0$); the concepts of GEEP and GEI coincide. GEEP is thus a generalization of the traditional GEI. With positive entry costs, a GEEP equilibrium differs from the traditional concept through two different channels. First, agents have different budget sets and endogenously make their participation decisions. Second, trading activities use some of society's resources and thus crowds out private consumption, as seen in the market clearing condition at date $t = 0$. This phenomenon, which we call the displacement effect, probably plays a minor role in actual economies. Extensions of our model could transfer a fraction of trading fees to certain consumers (such as exchange owners), or seek to provide a more detailed description of the financial industry.

We now examine the welfare properties of equilibrium with endogenous participation. As already seen in the GEI subcase, equilibrium allocations are usually Pareto inefficient because the absence of certain markets induces incomplete risk-sharing. With two periods and a single good, however, GEI allocations are known to satisfy a limited or constrained form of efficiency: no social planner can improve the utility of all agents when income transfers are constrained to belong to the asset span. This limited form of efficiency easily generalizes to our setting.

Definition 2. An allocation $\left(c_0^h; \tilde{c}^h\right)_{h \in H}$ is called feasible if and only if

1. For all h, there exists $\left(\theta_0^h; \theta^h\right) \in \mathbb{R} \times \mathbb{R}^J$ such that $\tilde{c}^h = \tilde{e}^h + \theta_0^h + \tilde{a} \cdot \theta^h$
2. $\int (c_0^h + 1_{\{\theta^h \neq 0\}}) d\mu(h) = \int e_0^h d\mu(h)$; and $\int \tilde{c}^h(\omega) d\mu(h) = \int \tilde{e}^h(\omega) d\mu(h)$ for all $\omega \in \Omega$.

We can now introduce

Definition 3. An allocation $\left(c_0^h; \tilde{c}^h\right)_{h\in H}$ is called constrained Pareto-efficient if it is feasible and if no other feasible allocation $(d_0^h; \tilde{d}^h)_{h\in H}$ is such that $U^h(d_0^h; \tilde{d}^h) > U^h(c_0^h; \tilde{c}^h)$ for all $h \in H$.

By standard arguments, it is now straightforward to show

Theorem 1. If $[\pi_0; \pi; \left(\theta_0^h; \theta^h; c_0^h; \tilde{c}^h\right)_{h\in H}]$ is a GEEP, the allocation $\left(c_0^h; \tilde{c}^h\right)_{h\in H}$ is constrained Pareto-efficient.

Proof. Assume that there exists a feasible allocation $(d_0^h; \tilde{d}^h)_{h\in H}$ such that $U^h\left(d_0^h; \tilde{d}^h\right) > U^h(c_0^h; \tilde{c}^h)$ for all h. We know that for all h; there exists $(\eta_0^h; \eta^h)$ such that $\tilde{d}^h = \tilde{e}^h + \eta_0^h + \tilde{a} \cdot \eta^h$: Since $(d_0^h; \tilde{d}^h)$ is strictly preferred to $(c_0^h; \tilde{c}^h)$; it must be that $d_0^h + \pi_0 \eta_0^h + \pi \cdot \eta^h + 1_{\{\eta^h \neq 0\}} > e_0^h$. We aggregate across consumers: $\int \eta_0^h d\mu = 0$; $\int \eta^h d\mu = 0$ and $\int (d_0^h + 1_{\{\eta^h \neq 0\}}) d\mu > \int e_0^h d\mu$; which contradicts feasibility. ■

The theorem implies that as in the GEI case, the introduction of a new asset cannot make all agents worse off.

2.1. Existence of Equilibrium

In a standard GEI economy, we know that an equilibrium always exists when there are two periods and a single good (Cass, 1984 ; Werner, 1985; Hens, 1991).[3] We now establish the existence of our GEEP equilibrium when the state space $\Omega = \{1; ...; S\}$ is finite and the economy satisfies standard hypotheses.

Assumption 1. The utility function U^h of every agent is continuous, strongly monotone and strictly quasi-concave on $\mathbb{R}^{S+1}_{++}$:

Despite this simplifying hypothesis on individual preferences, the existence proof is slightly more complicated in our model than in the GEI case. This is because at

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[3] There is an extensive literature on the existence of equilibrium in GEI economies with one good and two periods (Duffie and Shafer, 1985; Husseini, Lasry and Magill, 1990; Geanakoplos and Shafer, 1990; Hirsch, Magill and Mas-Colell, 1990). With more than one good or more than one period, the existence of equilibrium can only be obtained as a generic property (Hart, 1975; Geanakoplos and Polemarchakis, 1986).

prices where agents are indifferent between entry ($\theta^h \neq 0$) and no-entry ($\theta^h = 0$); individual demand for consumption and assets is a non-convex correspondence. In order to prove existence, we make the following hypothesis.

Assumption 2. There exists a finite number of individual types $h = 1, \ldots, H$, and a continuum of agents of each type.

We can then show

Theorem 2. Under Assumptions 1 and 2, there exists a GEEP equilibrium.

Proof. See Appendix ■

Given standard technical conditions, the proof of existence directly extends to any economy in which the measure μ of agents over types is atomless.

3. The CARA-Normal Economy with Endogenous Participation

In order to analyze the effect of financial innovation on investor participation and asset prices, we consider the tractable class of CARA-normal economies. In this section, we solve the participation and consumption-portfolio decision of an individual investor, and then derive the equations that jointly determine the participation structure and the prices of financial assets.

We consider a standard CARA-normal setup. Individual endowments and the payoffs of risky assets are jointly normal, and each investor h has separable utility

$$U(c_0, \tilde{e}) = -\frac{1}{\gamma}\left[\exp(-\gamma c_0) + \beta E \exp(-\gamma \tilde{e})\right]. \tag{3.1}$$

The assets $\{1, \tilde{a}_1, \ldots, \tilde{a}_J\}$ generate a linear subspace in the set $L^2(\Omega)$ of square-integrable random variables. Without loss of generality, we assume that the assets are orthonormal, implying $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_J) \sim N(0, I)$. We denote by A the span of risky assets $\{\tilde{a}_1, \ldots, \tilde{a}_J\}$, and by A_0 the span of all traded assets $\{1, \tilde{a}_1, \ldots, \tilde{a}_J\}$. Given a linear subspace V, it is convenient to represent by $\tilde{x}^V$ the projection on V of a random variable $\tilde{x}$, and by $V^{\perp}$ the subspace orthogonal to V.

3.1. Individual Entry Decision

This section analyzes the participation and consumption-portfolio decision of an individual investor h. We first calculate the consumption-portfolio choice under entry and no-entry in the risky asset market. Comparing the resulting utility levels yields the optimal entry decision.

When trading in risky assets, investor h maximizes $U\left(c_0^h; \tilde{c}^h\right)$ under the budget constraints $c_0^h + \pi_0\theta_0^h + \pi\cdot\theta^h + \quad \leq e_0^h$ and $\tilde{c}^h \leq \tilde{e}^h + \theta_0^h + \tilde{a}\cdot\theta^h$. It is convenient to consider the tradable portfolio $\tilde{m}^A \equiv -(R/\gamma)\sum_{j=1}^J \pi_j\tilde{a}_j$, that is determined by asset prices and the individual coefficient of risk aversion.

Theorem 3. When participating in the risky asset market, the investor buys

$$\theta_j^{h,p} = -\mathrm{Cov}(\tilde{a}_j, \tilde{e}^h) - R\pi_j/\gamma \tag{3.2}$$

units of risky asset j, and

$$\theta_0^{h,p} = \frac{R}{1+R}\left\{ e_0^h - \left[\left(E\tilde{e}^h - \gamma\,\mathrm{Cov}\left(\tilde{m}^A, \tilde{e}^{hA}\right) - \tilde{m}^A \right)/R + \gamma\left[\mathrm{Var}\left(\tilde{e}^{hA\perp}\right) + \mathrm{Var}\left(\tilde{m}^A\right)\right]/2 + \gamma^{-1}\ln(R\beta) \right] \right\} \tag{3.3}$$

units of the riskless asset. The corresponding consumption levels are then

$$c_0^{h,p} = \frac{1}{1+R}\left\{ R\left(e_0^h - \quad\right) + E\tilde{e}^h - \gamma\,\mathrm{Cov}\left(\tilde{m}^A; \tilde{e}^{hA}\right) - \tilde{m}^A - \gamma\left[\mathrm{Var}\left(\tilde{e}^{hA\perp}\right) + \mathrm{Var}\left(\tilde{m}^A\right)\right]/2 - \gamma^{-1}\ln(R\beta) \right\} \tag{3.4}$$

and $\tilde{\mathbf{c}}^{h,p} = E\left(\tilde{e}^h\right) + \theta_0^{h,p} + \tilde{m}^A + \tilde{e}^{hA\perp}_0$. The investor attains utility $U^p = -(1 + R^{-1})\exp(-\gamma c_0^{h,p})/\gamma$.

Proof. See Appendix ∎

The investor thus exchanges the marketable component $\tilde{e}^{hA}$ of her income risk for the tradable portfolio $\tilde{m}^A$, which allows an optimal allocation of risk and return.

We now examine the decision problem of an investor h trading only the riskless asset. The agent bears all her endowment risk in period 1, and the random consumption satisfies

$$\tilde{c}^{h,n} = \tilde{e}^h + \theta_0^{h,n}. \tag{3.5}$$

By a direct application of Theorem 3, the optimal decision is given by

Corollary 1. When not entering the risky asset market, the investor chooses

$$\theta_0^{h;n} = \frac{R}{1+R}\left(e_0^h - E\left(\tilde{e}^h\right) + \frac{\gamma}{2} Var\left(\tilde{e}^h\right) + \frac{1}{\gamma}\ln(R\beta)\right) \tag{3.6}$$

units of the riskless asset. The corresponding consumption and utility levels are then

$$c_0^{h;n} = \frac{1}{1+R}\left[Re_0^h + E\tilde{e}^h - \gamma Var\left(\tilde{e}^h\right)/2 - \gamma^{-1}\ln(R\beta)\right];$$

$\tilde{e}^{h;n} = \tilde{e}^h + \theta_0^{h;n}$, and $U^n = -(1+R^{-1})\exp(-\gamma c_0^{h;n})/\gamma.$

The investor participates in the risky asset market if $U^p > U^n$; or equivalently $c_0^{h;p} > c_0^{h;n}$. This leads to

Theorem 4. The investor trades risky assets if

$$\frac{\gamma}{2} Var\left(\tilde{e}^{hA} - \tilde{m}^A\right) > R; \tag{3.7}$$

and is indifferent to entry when this relation holds as an equality.

Proof. By Theorem 3 and Corollary 1, the condition $c_0^{h;p} > c_0^{h;n}$ is equivalent to

$$-R - \gamma Cov\left(\tilde{m}^A; \tilde{e}^{hA} - \tilde{m}^A\right) - \gamma Var\left(\tilde{m}^A\right)/2 > -\gamma Var\left(\tilde{e}^{hA}\right)/2;$$

which simplifies to $\gamma Var\left(\tilde{m}^A - \tilde{e}^{hA}\right)/2 > R.$ ■

Relation (3.7) has a simple geometric interpretation in $L^2(\Omega)$ that is illustrated in Figure 3.1. The agent trades risky assets if the distance between her income risk $\tilde{e}^{hA}$ and her optimal portfolio $\tilde{m}^A$ is larger than $\sqrt{2R/\gamma}$. Participation is negatively affected by a high interest rate because the entry cost is completely sustained in the first period, while the benefits from participation pertain to both periods.

3.2. Equilibrium

Let $P \subseteq H$ denote the subset of agents who participate in the risky asset market. Theorem 4 implies that

$$P_1 \subseteq P \subseteq P_1 \cup P_2;$$

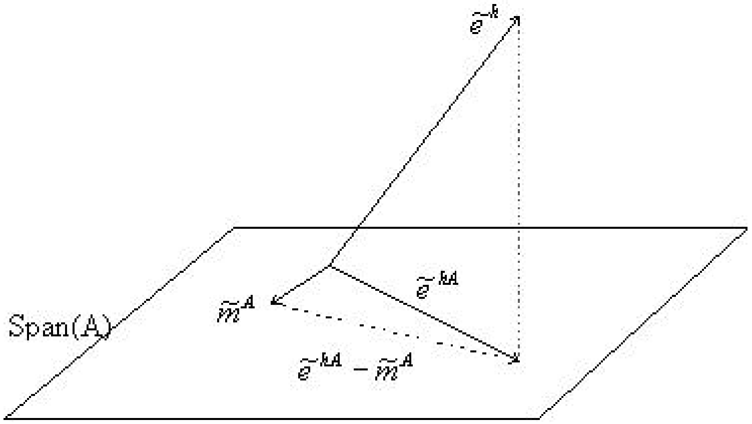

Figure 3.1: Geometric interpretation of the entry condition in $L^2(\Omega)$:

where $P_1 = \left\{ h \in H : \gamma Var(\tilde{e}^{hA} - \tilde{m}^A)/2 > R \right\}$ denotes the set of agents who strictly prefer participation, while $P_2 = \left\{ h \in H : \gamma Var(\tilde{e}^{hA} - \tilde{m}^A)/2 = R \right\}$ contains the agents who are indifferent to entry.

Market participants typically have different income risk characteristics than the entire population. We will show in Sections 5 and 6 that this difference is a driving element of the model.[4] It is convenient to denote by $e_0 = \int_H e_0^h d\mu(h)$ and

$$\tilde{e} = \int_H \tilde{e}^h d\mu(h)$$

the average income of the entire population. By contrast, we define the average endowment among participants

$$\tilde{e}^p = \int_P \tilde{e}^h d\mu^p(h);$$

where μ^p is the conditional measure $\mu/\mu(P)$ when $\mu(P) > 0$; and is equal to zero otherwise.

We obtain the price of risky assets by averaging (3:2) across participating agents:

$$\frac{\pi_j}{\pi_0} = -\gamma Cov(\tilde{e}^p; \tilde{a}_j):$$

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[4] Mankiw and Zeldes (1991) show the empirical pertinence of this distinction.

An asset is thus valuable if it is a good hedge against the aggregate income risk $\tilde{e}^p$. Since $\mathrm{Cov}(\tilde{a};\tilde{e}^p)=\mathrm{Cov}\left(\tilde{a};\tilde{e}^{pA}\right)$; the tradable component $\tilde{e}^{pA}$ of mean income corresponds to the market portfolio of the standard CAPM. By (3.2); each agent holds $\theta_j^h=\mathrm{Cov}\left(\tilde{e}^p-\tilde{e}^h;\tilde{a}_j\right)$ units of asset j in equilibrium, and the common portfolio $\tilde{m}^A$ satisfies

$$\tilde{m}^A=\tilde{e}^{pA}. \tag{3.8}$$

The optimal portfolio $\tilde{m}^A$ thus coincides with the mean tradable risk $\tilde{e}^{pA}$ of market participants.

Theorem 5. In equilibrium, the risky assets are worth

$$\pi_j=-\gamma\,\mathrm{Cov}(\tilde{e}^p;\tilde{a}_j)/R; \tag{3.9}$$

and the interest rate satisfies

$$\ln R=\ln R_0+\gamma\mu(P)+\frac{\gamma}{2}\int_P \mathrm{Var}\left(\tilde{e}^{hA}-\tilde{e}^{pA}\right)d\mu^p(h) \tag{3.10}$$

where $\ln R_0=\ln(1/\beta)+\gamma\left[E(\tilde{e})-e_0\right]-\gamma^2\int_H \mathrm{Var}\left(\tilde{e}^h\right)d\mu(h)/2.$

Proof. See Appendix $\blacksquare$

These pricing formulae hold whether the participation structure is exogenous or endogenous. For this reason, they coincide with the results obtained by Willen (1998) for GEI economies with exogenous participation.

When the participation set P is exogenous, it is straightforward to analyze the effect of participation on asset prices. An exogenous increase in P affects the riskless rate through two different channels, which correspond to the last two terms of equation (3.10). First, a higher participation implies that more first period resources $\mu(P)$ are absorbed in the entry process, leading to an increase the interest rate. Second, the interest rate is affected by the dispersion of income risk among participants. This effect can be interpreted by observing that

$$\int_P \mathrm{Var}\left(\tilde{e}^{hA}-\tilde{e}^{pA}\right)d\mu^p(h)=\int_P \mathrm{Var}\left(\tilde{e}^{hA}\right)d\mu^p(h)-\mathrm{Var}\left(\tilde{e}^{pA}\right):$$

When their tradable income risks are very dispersed, market participants use financial assets to sharply reduce the average variance of their second period consumption. This weakens their precautionary motive, reduces their marginal rates of substitution, and increases the equilibrium interest rate.

In our model of endogenous participation, asset prices and the set P are jointly determined. When the entry cost is infinite, no agent trades in the risky asset ($P = \emptyset$) and the equilibrium interest rate equals R_0. Conversely, consider an equilibrium in which no agent participates. The mean endowment of participants is zero ($e^p = 0$) and individual rationality imposes $\gamma Var\left(e^{hA}\right)/2 \leq R_0$. Let

$$ {}_{\max} = (\gamma/2R_0) \sup_{h \in H} Var\left(e^{hA}\right). $$

We can show

Proposition 1. When $\geq {}_{\max}$, the economy has a unique GEEP equilibrium, in which no agent trades in risky assets: $\mu(P) = 0$. On the other hand if $< {}_{\max}$, any GEEP equilibrium has a non-negligible set of participants.

Proof. Assuming $\geq {}_{\max}$, consider a GEEP equilibrium in which $\mu(P) > 0$. Equation (3.10) imposes that the equilibrium interest rate R is strictly greater than R_0. We infer that $\gamma Var\left(e^{hA}\right)/2 \leq {}_{\max} R_0 < R$, for all h and thus $P = \emptyset$.

Conversely, consider a GEEP equilibrium in an economy in which $< {}_{\max}$ and $P = \emptyset$. We know that there exists a (non-negligible) set S of agents such that $(\gamma/2R_0) Var\left(e^{hA}\right) >$. Therefore $\mu(P) \geq \mu(S) > 0$, which is a contradiction. ■

3.3. Effects of Financial Innovation

We can analyze the effect of financial innovation on asset prices from Theorem 5. When participation is exogenous, an increase in the asset span has no effect on the relative price $\pi_j/\pi_0 = -\gamma Cov(e^p, a_j)$ of preexisting assets. This result is consistent with the findings of Oh (1996) on CAPM economies. We also see that financial innovation also tends to increase the interest rate R, as in Elul (1997). This result has a simple intuition. A larger asset span allows greater risk sharing, leading to reduction in the precautionary motive and thus in the demand for the riskless asset. The interest rate is therefore higher in equilibrium.

In our GEEP model, participation is endogenous. Financial innovation can potentially change the market endowment e^p, and therefore the relative price π_j/π_0 of risky assets. The absolute price of an asset also depends on the interest rate. The variation of the interest rate is in general difficult to predict and depends on the possible change in e^p. In the following proposition, we show that the movement of the interest rate can be predicted when e^p remains constant as new assets are added to the economy.

Proposition 2. Let $\tilde{e}$ be a fixed random variable, and let R and P solve:

$$\begin{cases} P = \left\{ h : \frac{\gamma}{2} Var\left(\tilde{e}^{hA} - \tilde{e}^{A}\right) > R \right\} \\ \ln R = \ln R_0 + \gamma\,\mu(P) + \frac{\gamma}{2} \int_P \left[Var\left(\tilde{e}^{hA} - \tilde{e}^{A}\right) d\mu(h) \right] \end{cases}$$

Then R increases with financial innovation.

Proof. Financial innovation increases the assets span to $A' \supset A$; $A' \neq A$. The space A' can be decomposed in two orthogonal subspaces A and $B = A^{\perp} \cap A'$: By definition, R' and P' solve the system

$$\begin{cases} P' = \left\{ h : \frac{\gamma}{2} \left[Var\left(\tilde{e}^{hA} - \tilde{e}^{A}\right) + Var\left(\tilde{e}^{hB} - \tilde{e}^{B}\right)\right] > R' \right\} \\ \ln R' = \ln R_0 + \gamma\,\mu(P') + \frac{\gamma^2}{2} \int_{P'} \left[Var\left(\tilde{e}^{hA} - \tilde{e}^{A}\right) + Var\left(\tilde{e}^{hB} - \tilde{e}^{B}\right)\right] d\mu(h): \end{cases}$$

Assume that $R' < R$. The first equation implies $P \subseteq P'$; and we infer from the second equation that $R' \geq R$; a contradiction. ■

The proof has a straightforward intuition. Financial innovation and a decrease in the interest rate would both encourage entry. This would in turn create an increase in the interest rate, which leads to a contradiction. The proposition has the important implication that if the participants' average endowment does not vary, existing asset prices necessarily decrease with financial innovation. The heterogeneity of individual endowments thus plays a crucial role in determining the impact of financial innovation on the price of risky assets. In the next sections, we show this property by considering a multi-factor model of risk exposure.

4. A Factor Model

The heterogeneity of income risk can be conveniently specified when there exist a finite number L of risk factors $\{\varepsilon_1, ..., \varepsilon_L\}$: The factors correspond for instance to macroeconomic or sectoral shocks that affect the incomes of large groups of workers. We assume that the endowment of each investor $h \in H$ is a linear combination of the risk factors

$$\tilde{e}^h = E\tilde{e}^h + \sum_{\ell=1} \varphi^h_\ell \varepsilon_\ell; \tag{4.1}$$

and $\varphi^h = (\varphi^h_1, ..., \varphi^h_L)$ is called the individual vector of factor loadings. For simplicity, we do not consider idiosyncratic income shocks in the model.

We assume that the risk factors and the payoffs of financial assets are jointly normal. Without loss of generality, the factors are chosen to have unit variances and be mutually uncorrelated: $\tilde{\varepsilon} = \{\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_L\} \sim N(0, I)$. The measure μ on the investor set H induces a measure ν on the Euclidean space R^L of factor loadings. Since the distribution of the deterministic component of income does not affect equilibrium, we will neglect the difference between these two measures in the rest of this paper. The mean endowment of market participants can thus be written

$$\tilde{e}^p = E\tilde{e}^p + \sum_{\ell=1}^{L} \left(\int_P \phi_\ell \, d\mu^p \right) \tilde{\varepsilon}_\ell = E\tilde{e}^p + \sum_{\ell=1}^{L} \phi_\ell^p \tilde{\varepsilon}_\ell; \qquad (4.2)$$

where $\phi^p = (\phi_1^p, \ldots, \phi_L^p)$ denotes the average factor loadings of participants.

The factors generally do not belong to the span A of risky assets. It is therefore useful to consider their projections or tradable components $\tilde{\varepsilon}_\ell^A = \sum_{j=1}^J Cov(\tilde{\varepsilon}_\ell, \tilde{a}_j)\, \tilde{a}_j$. The variance-covariance matrix $\Sigma^A = Var\left(\tilde{\varepsilon}^A\right)$ has coefficients $Var\left(\tilde{\varepsilon}_\ell^A\right) = \sum_{j=1}^J [Cov(\tilde{\varepsilon}_\ell, \tilde{a}_j)]^2$ on the diagonal, and cross elements

$$Cov\left(\tilde{\varepsilon}_k^A, \tilde{\varepsilon}_l^A\right) = \sum_{j=1}^{J} Cov(\tilde{\varepsilon}_k, \tilde{a}_j)\, Cov(\tilde{\varepsilon}_l, \tilde{a}_j). \qquad (4.3)$$

Since Σ^A is positive semidefinite, the function $\|\phi\|_A = \sqrt{\phi' \Sigma^A \phi}$ defines a pseudo-norm of R^L. This notation is particularly useful because $Var(\tilde{e}^A) = \|\phi\|_A^2 = \phi' \Sigma^A \phi$ for any linear combination $\tilde{e} = \sum_{l=1}^L \phi_\ell \tilde{\varepsilon}_\ell$ of the factors.

Equations (4.1) and (4.2) thus imply that $Var\left(\tilde{e}^{hA} - \tilde{e}^{pA}\right) = \left\|\phi^h - \phi^p\right\|_A^2$. Equilibrium relation (3.10) can thus be rewritten

$$\ln R = \ln R_0 + \gamma \mu(P) \quad + \frac{\gamma}{2} \int_P \|\phi - \phi^p\|_A^2 \, d\mu^p(\phi) \Big]; \qquad (4.4)$$

while entry decision (3.8) becomes

$$\frac{\gamma}{2} \|\phi - \phi^p\|_A^2 > \quad R. \qquad (4.5)$$

The set $\{\phi \in R^L : (\gamma/2)\phi' \Sigma^A \phi \leq R\}$ thus contains the factor loadings of investors who do not trade risky assets. This set is an ellipsoid when the matrix Σ^A has full rank, and a cylinder when Σ^A is singular.

5. The One Factor Economy

We consider in this section an economy with a unique risk factor $\tilde{\varepsilon}$. Existing assets span only partially the risk $\tilde{\varepsilon}$, and the hedging coefficient

$$\alpha = Var(\tilde{\varepsilon}^A)$$

is a useful measure of market incompleteness. Since the risk factor $\tilde{\varepsilon}$ has unit variance, the coefficient α is contained between 0 and 1. The distribution of the factor loading φ is specified by a measure μ on the real line. It is convenient to consider

Assumption 3. The measure μ has density $f(\varphi)$ with respect to Lebesgue measure, and $Supp(f) = \{\varphi \in \mathbb{R} : f(\varphi) > 0\}$ is an unbounded interval.

Many of the results presented in this section can be generalized when the measure μ display singularities.

We first analyze the special cases $\alpha = 0$ or $\kappa = 0$: When assets have no correlation with the risk factor ($\alpha = 0$); the participation set is empty under costly entry ($\kappa > 0$), or indeterminate under free entry ($\kappa = 0$): In either case, asset prices are uniquely determined: $R = \underline{R}$ and there is no risk premium. When the hedging coefficient is positive ($\alpha > 0$) and the entry cost is positive and finite ($0 < \kappa < \infty$); we infer from Proposition 1 and Assumption 3 that the set of participants and non-participants both have a positive measure: $0 < \mu(P) < 1$:[5] Denote by $\overline{R}(\alpha)$ the interest rate corresponding to free entry:

$$\overline{R}(\alpha) = R_0 \exp[\alpha\gamma^2 Var_H(\varphi)/2]:$$

It is easy to show that[6]

$$\mu(P)(Var_P\, \varphi) \leq Var_H\, \varphi;$$

implying $R = R_0 \exp\{\gamma\mu(P)[\kappa + \alpha\gamma(Var_P\, \varphi)/2]\} \geq \overline{R}(\alpha) \exp[\gamma\kappa\mu(P)]$ in any equilibrium. Finally, we know that that $P = \emptyset$ when the entry cost is infinite. These results are summarized in Table 1.

---

[5] If everyone participates, $\varphi^p = \bar{\mu}$ and $\alpha\gamma(\varphi^h - \bar{\mu})^2/2 \geq \kappa R$ for almost every agent h. This is impossible since the density f is strictly positive on a neighborhood of $\bar{\mu}$:

[6] Since
$$\int_P (\varphi - \bar{\mu})^2 d\mu = \int_P (\varphi - \varphi^p)^2 d\mu + \mu(P)(\varphi^p - \bar{\mu})^2$$
we infer that $\mu(P) Var_P\, \varphi \leq \int_P (\varphi - \bar{\mu})^2 d\mu \leq Var_H(\varphi)$:

Table 1: Participation Structure

	= 0	2 (0; 1)	= 1
® = 0	P undetermined $R = \underline{R}$	P = ; $R = \underline{R}$; ' p = 0	P = ; $R = \underline{R}$; ' p = 0
® 2 (0; 1]	P = H $R = \overline{R}$(®); ' p = ¬	; ½ P ½ H $\underline{R} < R$	P = ; $R = \underline{R}$; ' p = 0

The participation set P for various levels of the hedging coe¢cient ® and the entry cost

We now further examine the non-degenerate case ® 2 (0; 1] and 2 (0; 1): The average endowment of the participants satis…es $\tilde{e}^p = \int_P$ ' d ¹ p ¡ ¢ ¨ = ' p ¨: The agents indi¤erent between entry and no-entry satisfy

$$\frac{°}{2} Var(\tilde{e}^{'A} ¡ \tilde{e}^{pA}) = \frac{®°}{2}(' ¡ '^p)^2 = R$$

This equation has two solutions ' = ' p § ¤ that lie at the same distance ¤ from ' p. The participation set is therefore P = (¡ 1 ; ' p ¡ ¤] [[' p + ¤; + 1): The indi¤erence condition can be rewritten as

$$\frac{°}{2} ®¤^2 = R \qquad (5.1)$$

By equation (4.4), the equilibrium interest rate satis…es

$$\ln R = \ln R_0 + °¹(P) \left[+ \frac{®°}{2} Var_P(') \right]; \qquad (5.2)$$

where $Var_P(') = \int_P '^2 d¹^p(') ¡ ('^p)^2$ is the variance of the participants' factor loadings. The de…nition of ' p also implies

$$\int_P (' ¡ '^p) d¹(') = 0; \qquad (5.3)$$

where $¹ = \int_R ' d¹(')$ is the aggregate risk in the economy. We have thus derived a system of three equations (5:1) ¡ (5:3) in the three unknowns (R; ' p; ¤).

We can …rst show

Theorem 6. There exists a unique GEEP equilibrium.

Proof. See Appendix. ¥

This result allows us to perform comparative statics of the GEEP equilibrium. We can thus analyze how …nancial innovation, i.e. an exogenous increase in ®; a¤ects the equilibrium values of the interest rate.

Proposition 3. The riskless rate R is increasing with financial innovation.

Proof. See Appendix. ■

As in models with exogenous participation, the introduction of new assets increases the riskless rate R: Improved risk sharing reduces the precautionary motive, reduces the demand for the riskless asset, and therefore increases the interest rate. [Add a word of explanation on GEEP economies].

Proposition 4. Financial innovation decreases φ_p if and only if

$$[1 - \alpha\gamma^2\mu(P)(Var_P\,\varphi)/2][f(\varphi_p + \delta) - f(\varphi_p - \delta)] < 0; \qquad (5.4)$$

and increases the fraction $\mu(P)$ of market participants if and only if the condition

$$\alpha\gamma^2\mu(P)(Var_P\,\varphi)/2 < 1 \qquad (5.5)$$

is satisfied.

Proof. See Appendix. ■

The measure $\mu(P)$ of market participants increases when the effect of innovation on the riskless rate due to the precautionary motive is not too strong. It is interesting to note that financial innovation may imply a decrease in participation. In a general GEEP model, this can occur because financial innovation changes the price of existing assets. Some agents may find the new prices sufficiently unattractive that they prefer to leave the market. In our CARA-normal model, the agents' decision is affected by the riskless rate and by movements in the market portfolio φ^p.

We now derive sufficient conditions that guarantee that the function $\varphi^p(\alpha)$ is decreasing in α: Consider

SC1. The density f is decreasing on $Supp(f) = [c, +\infty)$

SC2. $\gamma^2(Var_H\,\varphi)/2 < 1$

Under these conditions we know that $\alpha\gamma^2\mu(P)(Var_P\,\varphi)/2 \le \gamma^2(Var_H\,\varphi)/2 < 1$ and $f(\varphi_p + \delta) < f(\varphi_p - \delta)$ in any equilibrium. The function $\varphi^p(\alpha)$ is therefore decreasing in α: Densities f satisfying (SC1) include the exponential and $\chi^2(1)$ functions on $[0, \infty)$:

We can also find regions of the $(®;\ )$ plane where $d'^p=d® < 0$: Condition (5.5) is satisfied when $® < ®_1 = 2=(°^2 Var_P ')$; i.e. when markets are sufficiently incomplete. The next question is to control for $f('_p + ¤)$ ¡ $f('_p$ ¡ $¤)$: Consider

SC1a The density $f(')$ is decreasing on a neighborhood of $¹$:

For any $®$ ¸ 0; this condition imposes that $G(®;\ ) = f('_p + ¤)$ ¡ $f('_p$ ¡ $¤)$ is negative on a neighborhood of $\ = 0$: This implies $d'_p=d®\ \ 0$ on a neighborhood of $[0;®_1]$ £ $f0g$: Note that condition (SC1a) is satisfied by the density of a log-normal random variable Z, $\ln Z$ » $N(³;¾^2)$.[7] More generally, condition (SC1a) corresponds to densities skewed to the right. We note that the assumptions of small entry costs and right skewness are very appealing for empirical applications.

To illustrate this effect, consider an asset **e** with a positive expected payoff E**e**: In equilibrium, the asset is worth:

$$\frac{1}{R}[E\mathbf{e}\ ¡\ °Cov(\mathbf{e};e^p)] = \frac{1}{R}[E\mathbf{e}\ ¡\ °'^pCov(\mathbf{e};\ddot{-})]$$

Innovation increases the riskless rate, and therefore tends to reduce the price of a risky asset. This result is well-known in GEI models with exogenous participation, and seems contradictory with the empirical evidence that the price of existing assets tends to increase following the introduction of new assets (). When participation is endogneous, however, we note that the mean factor $'^p$ of market participants can vary with financial innovation and can therefore potentially reverse the interest rate effect. More specifically, consider an asset **e** that is positively correlated with the risk factor $\ddot{-}$: Financial innovation increases the relative price

$$\frac{¼(\mathbf{e})}{¼_0} = E\mathbf{e}\ ¡\ °'^pCov(\mathbf{e};\ddot{-})$$

if $'^p$ decreases with innovation.

Variations in the mean risk $'^p$ occur when new entrants tend to be less exposed to the factor than the original market participants. This effect is thus closely related to asymmetries in the distribution of risk in the population, as is now shown.

Proposition 5. If the measure $¹$ is symmetric, financial innovation does not affect the mean endowment $'^p$ of market participants, and the relative price $¼(\mathbf{e})=¼_0$ of existing assets.

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[7] In this case, we know the mean $\bar{-} = \exp(³ + ¾^2=2)$ and f reaches a maximum at $'_{max} = \exp(³$ ¡ $¾^2) < \bar{-}$:

Proof. Consider the function

$$G(') = Z_{¡1}^{'_p ¡ ¤} (' ¡ '_p)d¹ + Z_{'_p+¤}^{+1} (' ¡ '_p)d¹:$$

We observe that the function $G(')$ is strictly decreasing and satisfies $G(\bar{+}) = 0$: Equation (5.3) implies that $G('_p) = 0$: Therefore $'^p = ¹$. The only effect that innovation has on asset prices is through the interest rate R. ¥

An asymmetric distribution of risk in the economy is therefore required in order to obtain movements in the mean risk factor $'^p$:

The market portfolio $E\tilde{e}^p + '^{p\ddot{\sim}A}$ is worth

$$P_m = \frac{1}{R} £ E\tilde{e}^p ¡ ®°('^p)^2 ¤ : \tag{5.6}$$

We assume without loss of generality that $¹ > 0$: Skewness can manifest in two ways, either there is a majority of agents with a low exposure to the risk factor, and few with a large exposure to it, or the opposite. As can be seen in the appendix (in the formulas used to prove proposition 4), it is for the first type that innovation results in a decrease of $'^p$ and therefore an upward pressure on asset prices. We find this result encouraging as it appears to be the case that in reality risks are distributed in a asymmetric way with this kind of skewness.[8]

Proposition 6. Financial innovation increases the price of the market portfolio if

$$\frac{E\tilde{e}^p ¡ P_m}{P_m}(¡²_{'_p;a}) > ²_{R;®}$$

Proof. We infer from equation 5.6 that

$$\frac{dP_m}{d®} = ¡\frac{P_m}{R^2}\frac{dR}{d®} ¡ \frac{2°}{R}'_p\frac{d'^p}{d®} = \frac{P_m}{®R} ¡²_{R;®} ¡ \frac{E\tilde{e}^p ¡ P_m}{P_m}²_{'_p;®} ¸ ;$$

which gives the desired result. [This formula is wrong] ¥

This relation tells us by how much the effect of innovation on participation has to be stronger that the effect on the riskless rate in order for innovation to increase

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[8] There are few firms that have a large exposure to risk issuing stock that is bought by a multitude of investors that have a low exposure to risk.

asset prices in general. A number of conditions must be satisfied to observe this effect. We need the effect on the riskless rate to be moderate, this helps in two ways, by making the effect of innovation on φ^p stronger, and by moderating the increase of the discounting factor $1/R$. Then we need that risk in the economy be distributed with enough skewness to have a large (negative) effect on φ^p. As we now show, the conditions for an effect on the risk premium, measured by the ratio of the return for the market portfolio to the return of the risk free rate, are less stringent.

Proposition 7. The risk premium in our economy is larger than in a full participation economy if the condition

$$\varphi^p > \mu$$

is satisfied for all financial structures.

Proof. Let RP and RP_0 respectively denote the risk premium in the restricted and full ($= 0$) participation economies. We know that

$$\begin{aligned} RP &= 1 - \left(\frac{\gamma \varphi^p \mathrm{Cov}(\tilde{e}^{pA}; \tilde{\theta})}{E\,\tilde{e}^p}\right)^{-1}; \\ RP_0 &= 1 - \left(\frac{\gamma \mu \mathrm{Cov}(\tilde{e}^{pA}; \tilde{\theta})}{E\,\tilde{e}^p}\right)^{-1}; \end{aligned}$$

and therefore $RP > RP_0$ if and only if $\varphi^p > \mu$. ■

Note that a necessary condition to get this result is that the distribution of risk is skewed with a majority of agents having a low exposure to the risk factor. This result is consistent with that found in the literature of asset pricing with restricted participation[9], were it is shown that restricted participation biases the risk premium upwards, thus helping to explain the equity premium puzzle.

6. The Two Factor Economy

We now consider an economy with two risk factors $\tilde{\theta}_1$ and $\tilde{\theta}_2$. We assume for simplicity that no asset is jointly correlated to both factors. Equation (4.3) implies that the variance-covariance matrix Σ^A is diagonal:

$$\Sigma^A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix};$$

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[9] See Basak and Cuoco (1998), or Vissing-Jorgensen (1998).

where $\alpha_\ell = Var\left(\tilde\varepsilon_\ell^A\right)$, $\ell = 1, 2$. Financial innovation is formally described by changes in α_ℓ, $\ell = 1, 2$. Higher α_ℓ means better spanning of factor ℓ but unchanged spanning for the remaining factor.

We assume that the measure μ has a continuous density $f(\varphi_1, \varphi_2)$ with respect to Lebesgue measure. The equilibrium conditions can be rewritten

$$\begin{cases} P = \left\{(\varphi_1, \varphi_2) : \frac{\gamma}{2}\left[\alpha_1(\varphi_1 - \varphi_1^p)^2 + \alpha_2(\varphi_2 - \varphi_2^p)^2\right] \geq R\right\} \\ \ln R = \ln R_0 + \gamma\mu(P) + \frac{\gamma^2}{2}\int_P \left[\alpha_1(\varphi_1 - \varphi_1^p)^2 + \alpha_2(\varphi_2 - \varphi_2^p)^2\right] f(\varphi_1, \varphi_2)\, d\varphi_1 d\varphi_2 \\ \varphi_\ell^p = \frac{1}{\mu(P)}\int_P \varphi_\ell f(\varphi_1, \varphi_2)\, d\varphi_1 d\varphi_2, \quad \ell = 1, 2 \end{cases} \tag{6.1}$$

The factor loadings of participants are therefore located outside the ellipse

$$\frac{\gamma}{2}\left[\alpha_1(\varphi_1 - \varphi_1^p)^2 + \alpha_2(\varphi_2 - \varphi_2^p)^2\right] = R,$$

which is centered on the participants' average factor loadings φ^p. The price of an existing asset $\tilde{a}$ correlated with factor k is given by

$$\pi(\tilde{a}) = \frac{1}{R}\left[E\tilde{a} - \gamma Cov(\tilde{a}, \tilde{e}^p)\right] = \frac{1}{R}\left[E\tilde{a} - \gamma\varphi_k^p Cov(\tilde{\varepsilon}_k, \tilde{a})\right]. \tag{6.2}$$

We can analyze in the (φ_1, φ_2) space how participation is affected by financial innovation. A higher interest rate widens the ellipse vertical and horizontal axes whereas higher factor one spanning (higher α_1) translates into a reduced length of the ellipse horizontal axis.

The following proposition focus on the effect of financial innovation when f is symmetric.

Proposition 8. Let f be symmetric around its mean μ. Then $\varphi^p = \mu$ for any financial structure, and R is increasing with financial innovation.

Proof. See Appendix. $\blacksquare$

Figure 6.1 illustrate Proposition 2 in the case of increasing spanning of risk factor one. Before financial innovation only the agents outside the wider ellipse trade in risky assets. After financial innovation, all agents with factor loadings in the light grey area enter risky asset markets whereas the agents in the dark area exit. The participants average factor loadings remain equal to the means of distribution f and, as a direct consequence of Proposition 1, the interest rate increases widening the ellipse vertical axis.

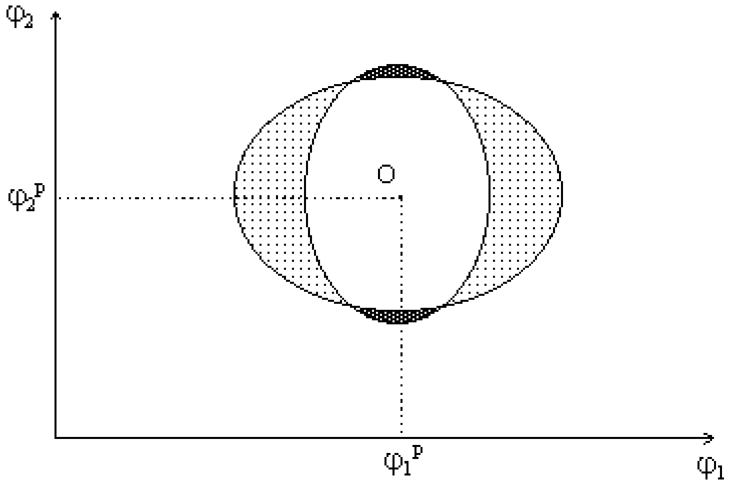

Figure 6.1:

In general financial innovation (higher α_1) changes the set of participants' factor loadings symmetrically around the ellipse center. When f is symmetric, the same measure of agents with opposite risk exposure enter and exit the economy so that the participants' average factor loadings do not change.

We now consider the case in which f is skewed towards the origin. In Figures (6.2) − (6.4); a representative simulation results are reported. Financial innovation relative to risk factor one increases participation and the interest rate. The price of existing assets will be affected differently depending on their correlation with factor one. The increase in R will tend to reduce prices of all existing asset whereas the decline of both φ_1^p and φ_2^p will work in the opposite direction for assets positively correlated with risk factors. As figure 6.3 shows, this effect is higher for the assets in the direction of innovation.

In figure 6.5 we represent the simulation results in the space of factor loadings. With f skewed towards the origin, more agents with low risk exposure enter financial markets thereby reducing participants' average loading to both factor one and two (the ellipse center moves to O'). Despite the fact that financial innovation pertains only factor one and that risk factors are uncorrelated, φ_2^p declines as well. This is possible since a positive measure of agents exposed to risk factor two find convenient to participate to risky asset markets. Yet, entry will occur mainly in the direction of innovation making the effect on φ_1^p stronger.

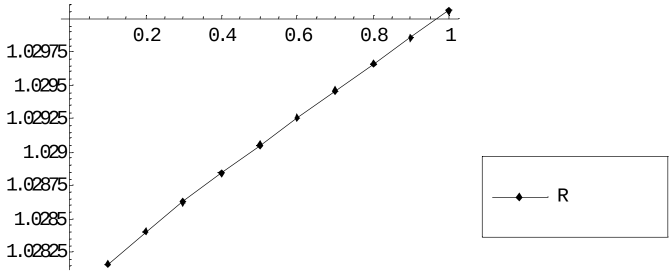

Figure 6.2: R plotted against ®$_1$:

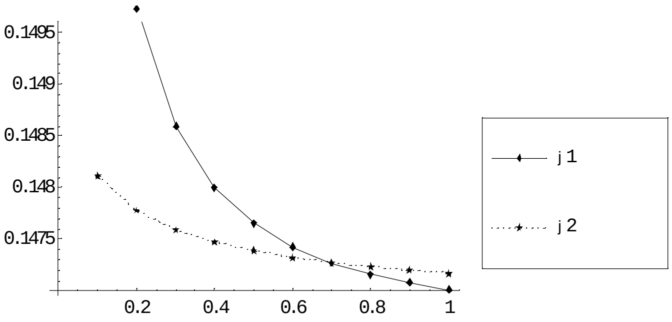

Figure 6.3: '$^p_1$; '$^p_2$ plotted against ®$_1$:

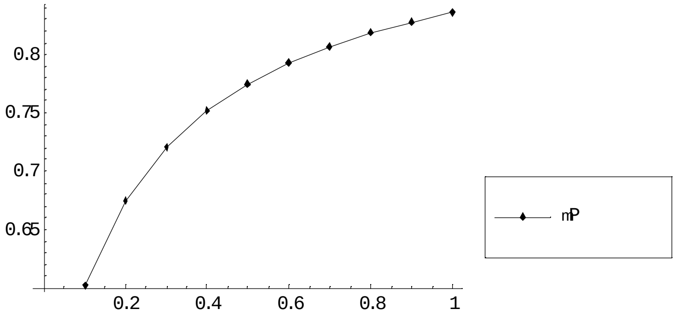

Figure 6.4: ¹(P) plotted against ®$_1$: ®$_2$ = 0:7, ° = 1, ¯ = 0:95; = 0:0005, R$_0$ = 1:02641, f = f$_1$ ¢ f$_2$; where f$_1$ = f$_2$ is log-normal with normal mean ¡ 2 and normal variance 0:4.

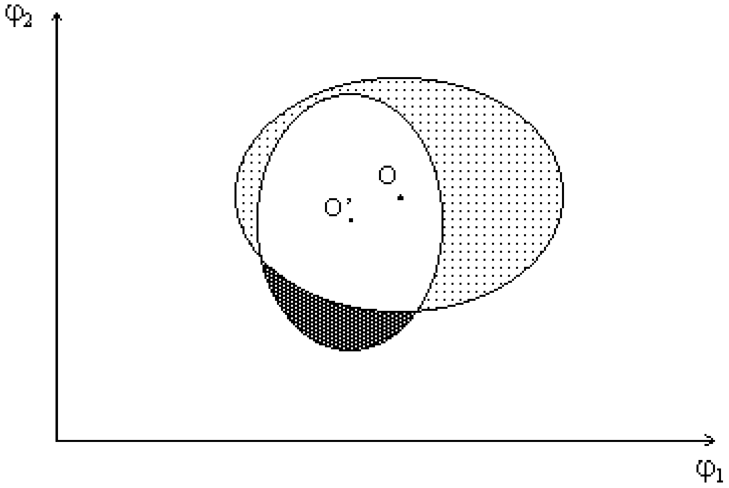

Figure 6.5:

The effect on the interest rate is positive in the simulation outlined in figure 6.2-6.4. Both the displacement and the precautionary savings effects happen to be positive. Yet we have obtained simulations in which the number of participants decrease creating a downward pressure in the interest rate via the displacement affect. We also conjecture that the precautionary saving effect could be negative as well. The intuition being that, within the class of participating agents, those with average exposure to risk factor one and with little exposure to risk factor two have higher precautionary motive after financial innovation.

7. Appendix

7.1. Proof of Theorem 2

We base our argument on the existence proof provided by Hens (1991) for the standard GEI case.

Individual Excess Demand

Given $p_0 > 0$ and a vector (π_0, π) of asset prices, it is convenient to define $q = (p_0, \pi_0, \pi)$ and the budget set

$$\widehat{B}^h(q) = \left\{ \left(c_0, \theta_0^h, \theta^h \right) : p_0(c_0 + 1_{\{\theta^h \neq 0\}}) + \pi_0\theta_0^h + \pi \cdot \theta^h \leq p_0 e_0^h \right\}.$$

The no-arbitrage set

$$Q = \left\{ (p_0, \pi_0, \pi) \in R_{++} \times R^{J+1} \;\middle|\; \begin{array}{l} \text{there exists } \lambda \in R^S_{++} \text{ such that} \\ \pi_j = \lambda . a_j \text{ for all } j = 0, \ldots, J \end{array} \right\}.$$

is an open convex cone of R^{J+2}, and it is useful to consider its closure

$$\overline{Q} = \left\{ (p_0, \pi_0, \pi) \in R_{+} \times R^{J+1} \;\middle|\; \begin{array}{l} \text{there exists } \lambda \in R^S_{+} \text{ such that} \\ \pi_j = \lambda . a_j \text{ for all } j = 0, \ldots, J \end{array} \right\}.$$

Given $q \in Q$, we can calculate the excess demands $Z^{hi}(q) \equiv [c_0^{hi}(q) + {} - e_0^h, \theta_0^{hi}(q), \theta^{hi}(q)]$ and $Z^{hn}(q) \equiv [c_0^{hn}(q) - e_0^h, \theta_0^{hn}(q), \theta^{hn}(q)]$ of a participating and non-participating agent of type h. Given a participation decision $d \in \{i, n\}$, the excess demand function $Z^{hd}(q)$ is continuous, homogeneous of degree 0, and satisfies Walras' law. We can then define the excess demand correspondence

$$Z^h(q) = \begin{cases} Z^{hi}(q) & \text{if } V^h\left[Z^{hi}(q)\right] > V^h\left[Z^{hn}(q)\right] \\ Z^{hn}(q) & \text{if } V^h\left[Z^{hi}(q)\right] < V^h\left[Z^{hn}(q)\right] \\ [Z^{hi}(q), Z^{hn}(q)] & \text{if } V^h\left[Z^{hi}(q)\right] = V^h\left[Z^{hn}(q)\right] \end{cases};$$

where $V^h(z)$ denotes the utility $U^h(c_0, e^h + \theta_0 + e.\theta)$ associated to an excess demand strategy $z = [c_0 + 1_{\{\theta \neq 0\}} - e_0^h, \theta_0, \theta]$. We observe that $Z^h(q)$ is homogeneous of degree 0, upper hemi-continuous and satisfies Walras' law.

Consider a vector $\overline{q} \in \overline{Q} \setminus Q$, $\overline{q} \neq 0$, and a sequence $\{q^n\}_{n=1}^{\infty}$ of elements of Q converging to $\overline{q}$. We want to show that $\inf\{\|z\|, z \in Z^h(q^n)\} \to \infty$. Proceed by contradiction and assume that there exists a bounded sequence $\{z^{n_k}\}_{k=0}^{\infty}$, $z^{n_k} \in$

$Z^h(q^{n_k})$ for all k. The sequence $\{z^{n_k}\}_{k=0}^{\infty}$ has then a cluster point $\bar{z}$. Without loss of generality, it is convenient to henceforth neglect subsequence notation and directly assume that $z^n \to \bar{z}$. Given $x \in \hat{B}^h(\bar{q})$, we know that x is the limit of a sequence $\{x^n\}$, $x^n \in \hat{B}^h(q^n)$. Since $x^n \in \hat{B}^h(q^n)$, we know that $V^h(x^n) \leq V^h(z^n)$ for all n. Letting n go to infinity, we infer that $V^h(x) \leq V^h(\bar{z})$ for all $x \in \hat{B}^h(\bar{q})$, which is absurd. This establishes that $\inf\{\|z\|; z \in Z^h(q^n)\} \to \infty$ as $n \to \infty$. We can also consider the matrices $M = [a_0, \dots, a_J]$ and $N = \begin{pmatrix} 1 \\ M \end{pmatrix}$; and show by a similar argument that $\inf\{\|z\|; z \in NZ^h(q^n)\} \to \infty$ as $n \to \infty$. Moreover since consumption is non-negative, the set $NZ^h(q^n) \geq -e^h$ is bounded below.

Market Excess Demand

We now define the market excess demand

$$Z(q) \equiv \sum_{h=1}^{H} \nu(h) Z^h(q).$$

The correspondence $Z(q)$ is upper hemi-continuous, convex and compact-valued, homogeneous of degree 0 and satisfies Walras' law: $q.Z(q) \equiv 0$. Moreover consider an arbitrary vector $\hat{q} \in Q$ and a sequence $\{q^n\}_{n=1}^{\infty}$ of elements of Q converging to a vector $\bar{q} \in \bar{Q} \setminus Q$, $\bar{q} \neq 0$. Since each $NZ^h(q^n)$ is bounded below, we infer that $NZ(q^n)$ is bounded below and $\inf\{\|z\|; z \in NZ(q^n)\} \to \infty$. The absence of arbitrage implies that $\hat{q} = N'\hat{\lambda}$ for some $\hat{\lambda} \in \mathbb{R}^{S+2}_{++}$. Since $\inf\{\|z\|; z \in NZ(q^n)\} \to \infty$, we infer that $\hat{q}.Z(q^n) = \hat{\lambda}.NZ(q^n) > 0$ for n large enough. We then conclude by standard arguments (Debreu, 1956; Grandmont, 1977; Hens, 1991) that there exists an equilibrium price.

7.2. Proof of Theorem 3

Let $u(c) = -\exp(-\gamma c)/\gamma$. The decision problem consists of maximizing

$$u(c_0) + \beta E u\left[\tilde{e}^h + \tilde{a}.\theta + R(e_0^h - c_0 - \pi.\theta - \quad)\right]$$

with respect to the unconstrained variables c_0 and θ. Since $\tilde{a}$ is normally distributed, we infer that

$$Eu(\tilde{c}) = u\left[E\tilde{e}^h + R(e_0^h - c_0 - \pi.\theta - \quad)\right] \exp\left[\gamma^2 Var(\tilde{e}^h + \tilde{a}.\theta)/2\right].$$

The objective function can thus be rewritten

$$u(c_0) + \beta u(D - Rc_0)\exp\left[\gamma R(\pi.\theta + \;) + \gamma^2 Var(\tilde{e}^{hA} + \tilde{a}.\theta)/2\right];$$

where $D = E\tilde{e}^h + Re_0^h - \gamma Var(\tilde{e}^{hA\star})/2.$

The utility maximization problem is decomposed in two steps. First, we find the optimal portfolio θ^h by minimizing the quadratic function

$$R\pi.\theta + \gamma Var(\tilde{e}^{hA} + \tilde{a}.\theta)/2;$$

that is equal to $R\sum_{j=1}^{J}\pi_j\theta_j + \frac{\gamma}{2}\left[Var(\tilde{e}^{hA}) + 2\sum_{j=1}^{J}\theta_j Cov(\tilde{e}^{hA};\tilde{a}_j) + \sum_{j=1}^{J}\theta_j^2\right].$ The first order condition implies that θ_j^h satisfies (3.2). The optimal portfolio θ^h is thus worth

$$\begin{aligned}\pi.\theta^h &= \sum_{j=1}^{J}\pi_j\left[-Cov(\tilde{a}_j;\tilde{e}^h) - R\pi_j/\gamma\right]\\ &= (\gamma/R)Cov\left(\tilde{m}^A;\tilde{e}^h\right) - (\gamma/R)Var(\tilde{m}^A)\\ &= \gamma Cov\left(\tilde{m}^A;\tilde{e}^{hA} - \tilde{m}^A\right)/R;\end{aligned}$$

and second period consumption satisfies $\tilde{e}^{hA} = \tilde{e}^{hA} + \tilde{a}.\theta^h = \tilde{m}^A.$

Second, we solve for optimal initial consumption c_0 by maximizing

$$u(c_0) + \beta u(D - Rc_0)\exp\left\{\gamma \; R + \gamma^2[Cov(\tilde{m}^A;\tilde{e}^{hA} - \tilde{m}^A) + Var(\tilde{m}^A)/2]\right\}:$$

We infer from the first order condition that

$$u'(c_0) = \beta R u'(D - Rc_0)\exp\left\{\gamma \; R + \gamma^2[Cov(\tilde{m}^A;\tilde{e}^{hA} - \tilde{m}^A) + Var(\tilde{m}^A)/2]\right\};$$

or equivalently $c_0 = -\ln(\beta R)/\gamma + D - Rc_0 - \; R - \gamma[Cov(\tilde{m}^A;\tilde{e}^{hA} - \tilde{m}^A) + Var(\tilde{m}^A)/2];$ which implies (3.4). We then deduce θ_0^h from the budget constraint. Since $u = -u'/\gamma;$ the optimal utility level satisfies

$$U^p = -[u'(c_0^h) + \beta E u'(\tilde{e}^h)]/\gamma = -(1 + 1/R)u'(c_0^h)/\gamma;$$

and thus $U^p = (1 + 1/R)u(c_0^h).$

7.3. Proof of Theorem 5

By (3.3) and (3.6), the mean demand $\int_H \theta_0^h d\mu(h)$ for the riskless asset is

$$\frac{R}{1+R}\left[\begin{array}{l} e_0 - E(\tilde{e}) - \mu(P) + \frac{\gamma}{2}\int_P [Var\left(\tilde{e}^{hA\star}\right) + Var\left(\tilde{e}^{pA}\right)]d\mu(h) \\ \quad + \frac{\gamma}{2}\int_{H\setminus P} Var\left(\tilde{e}^{h}\right) d\mu(h) + \gamma^{-1}\ln(R\beta) \end{array}\right].$$

In equilibrium, the interest rate therefore satisfies

$$\begin{aligned}\gamma^{-1}\ln(R\beta) &= E(\tilde{e}) - e_0 + \mu(P) - \frac{\gamma}{2}\int_H Var\left(\tilde{e}^h\right) d\mu(h) \\ &\quad + \frac{\gamma}{2}\int_{H\setminus P}\left\{Var\left(\tilde{e}^{hA}\right) - Var\left(\tilde{e}^{pA}\right)\right\} d\mu(h);\end{aligned}$$

which is equivalent to (3.10).

7.4. Proof of Theorem 6

For any $(\varphi_p;\delta) \in \mathbb{R} \times [0;+\infty)$; consider the function

$$G(\varphi_p;\delta) = \int_{-\infty}^{\varphi_p-\delta}(\varphi - \varphi_p)d\mu + \int_{\varphi_p+\delta}^{+\infty}(\varphi - \varphi_p)d\mu.$$

For every $\delta \geq 0$; the function $G_\delta(\varphi_p) \equiv G(\varphi_p;\delta)$ is continuous and strictly decreasing. It also satisfies $G_\delta(\varphi_p) \to +\infty$ as $\varphi_p \to -\infty$; and $G_\delta(\varphi_p) \to -\infty$ as $\varphi_p \to +\infty$. Therefore the equation $G(\varphi_p;\delta) = 0$ has a unique solution, which we denote $\varphi_p(\delta)$. By the Implicit Function Theorem, the function $\varphi_p(\delta)$ is differentiable. Let $\Delta(\mu) = f(\varphi^p+\delta) - f(\varphi^p-\delta)$ and $r(\mu) = f(\varphi^p+\delta) + f(\varphi^p-\delta)$. We observe that $\partial G/\partial\varphi_p = -\delta r - \mu(P) < 0$, $\partial G/\partial\delta = -\delta\Delta$; and therefore

$$\frac{d\varphi_p}{d\delta} = -\frac{\delta\Delta(\mu)}{\delta r + \mu(P)}.$$

The sign of $d\varphi_p/d\delta$ thus depends on the value of the density f at the endpoints $\varphi^p - \delta$ and $\varphi^p + \delta$. Since $\left|d\varphi_p/d\delta\right| \le \delta r/[\delta r + \mu(P)] < 1$; the functions $\varphi^p(\delta) - \delta$ and $\varphi^p(\delta) + \delta$ are respectively decreasing and increasing in δ. We also note that the monotonicity of these functions is strict.

We now consider the correspondence

$$P(\delta) = (-\infty;\varphi^p(\delta) - \delta] \cup [\varphi^p(\delta) + \delta;+\infty)$$

and the functions $H_0(¤) = ¹[P(¤)]$ and $H_1(¤) = ¹[P(¤)](Var_{P(¤)}')$. We want to show that the functions H_0 and H_1 are decreasing in ¤: First observe that the monotonicities of $'^p(¤)$ ¡ ¤ and $'^p(¤) + ¤$ implies that

$$H_0(¤) = \mathrm{Z}_{¡1}^{'^p(¤)¡¤} d¹ + \mathrm{Z}_{'^p(¤)+¤}^{+1} d¹$$

is strictly decreasing in ¤: It also straightforward that

$$\frac{dH_0}{d¤} = µ\frac{d'^p}{d¤} ¡ 1¶ f('^p ¡ ¤) ¡ µ\frac{d'^p}{d¤} + 1¶ f('^p + ¤)$$

and therefore

$$\frac{dH_0}{d¤} = ¡\frac{¤(r^2 ¡ ¢^2) + r¹(P)}{¤r + ¹(P)} < 0;$$

since $r^2 ¡ ¢^2 = 4f('^p ¡ ¤)f('^p + ¤) > 0$: Similarly, the function

$$H_1(¤) = \mathrm{Z}_{¡1}^{'^p¡¤} (' ¡ '_p)^2 f(')d' + \mathrm{Z}_{'^p+¤}^{+1} (' ¡ '_p)^2 f(')d'$$

has derivative

$$\frac{dH_1}{d¤} = ¤^2 \ f('_p ¡ ¤)\frac{d('_p ¡ ¤)}{d¤} ¡ f('_p + ¤)\frac{d('_p + ¤)}{d¤}^{¸} + \mathrm{Z}_P 2(' ¡ '_p)f(')d'$$

$$= ¤^2\frac{dH_0}{d¤} < 0:$$

The functions H_0 and H_1 are thus decreasing in ¤:

In equilibrium, R and ¤ are determined by the system

$$½ \begin{array}{l} \ln R = \ln R_0 + \ °H_0(¤) + ®°^2H_1(¤)=2 \\ R = °®¤^2=2: \end{array}$$

In the (¤; R) plane, these relations define two curves (C_1) and (C_2). We can write $R = C_1(¤)$ and $R = C_2(¤)$: We note that C_1 is decreasing, C_2 is increasing, $C_1(0) > R_0 > C_2(0)$; and $C_2(+1) = +1$: Therefore there exists a unique equilibrium. Moreover, we observe that an exogenous increase in ® pushes up both curves, leading to an increase in the interest rate R: The variation in ¤ depends on the relative slopes of H_0 and H_1; and is examined in greater detail in the proof Proposition 3. Similarly, an exogenous increase in the entry cost pushes up (C_1) and pushes down (C_2); leading to an increase in ¤: The variation of R seems ambiguous and is examined in greater detail in the proof of Proposition 3.

7.5. Proof of Proposition 3

We know that the equilibrium vector (R, Λ) solves the system

$$\begin{cases} R - \alpha\gamma\Lambda^2/2 = 0 \\ \ln R - \ln R_0 - \tau\gamma H_0(\Lambda) - \alpha\gamma^2 H_1(\Lambda)/2 = 0. \end{cases}$$

where $P = (-\infty, \varphi^p - \Lambda] \cup [\varphi^p + \Lambda, +\infty)$. The Jacobian matrix satisfies

$$J = \begin{pmatrix} & -\alpha\gamma\Lambda \\ R^{-1} & J_{22} \end{pmatrix} \tag{7.1}$$

where $J_{22} = -\tau\gamma H_0'(\Lambda) - \alpha\gamma^2 H_1'(\Lambda)/2 = -\gamma\tau(1+R)H_0'(\Lambda) > 0$. We infer that $\det J > 0$.

We now use Cramer's rule to find the effect of financial innovation on the equilibrium interest rate

$$\frac{dR}{d\alpha} = -\frac{1}{\det J}\begin{vmatrix} -\gamma\Lambda^2/2 & -\alpha\gamma\Lambda \\ -\gamma^2 H_1(\Lambda)/2 & J_{22} \end{vmatrix} > 0 \tag{7.2}$$

and the width parameter Λ:

$$\begin{aligned} \frac{d\Lambda}{d\alpha} &= -\frac{1}{\det J}\begin{vmatrix} & -\gamma\Lambda^2/2 \\ R^{-1} & -\gamma^2 H_1(\Lambda)/2 \end{vmatrix} \\ &= -(\tau/\alpha)\left[1 - \alpha\gamma^2\mu(P)(Var_P\,\varphi)/2\right]/\det J. \end{aligned}$$

We infer that

$$\frac{d\varphi^p}{d\alpha} = \frac{d\varphi^p}{d\Lambda}\frac{d\Lambda}{d\alpha}$$

has the same sign as $[f(\varphi^p + \Lambda) - f(\varphi^p - \Lambda)][1 - \alpha\gamma^2\mu(P)(Var_P\,\varphi)/2]$.

We can similarly analyze the effect of the transaction cost τ. We note that

$$\frac{dR}{d\tau} = -\frac{1}{\det J}\begin{vmatrix} R & -\alpha\gamma\Lambda \\ -\gamma H_0(\Lambda) & J_{22} \end{vmatrix}$$

has the sign of $\alpha\gamma^2\Lambda\mu(P) - RJ_{22}$, while

$$\frac{d\Lambda}{d\tau} = -\frac{1}{\det J}\begin{vmatrix} & R \\ R^{-1} & -\gamma H_0(\Lambda) \end{vmatrix} > 0.$$

This implies that the mass of participants decreases with the transaction cost τ. Finally, $d\varphi^p/d\alpha$ has the sign of $f(\varphi^p + \Lambda) - f(\varphi^p - \Lambda)$.

7.6. Proof of Proposition 4

By definition of ¹(P); the effect of financial innovation on participation is given by

$$\frac{d¹(P)}{d®} = \frac{@¹(P)}{@'^p}\frac{d'^p}{d®} + \frac{@¹(P)}{@¤}\frac{d¤}{d®} = ¡\ ¢(¹)\frac{d'^p}{d®}\ ¡\ r(¹)\frac{d'^p}{d¤}:$$

Plug in the derivatives $d'^p{=}d®$ and $d'^p{=}d¤$ calculated in the proof of Proposition 3:

$$\frac{d¹(P)}{d®} = \frac{}{®\det J}\ 1\ ¡\ \frac{®°^2}{2}¹(P)(Var_P\ ')\ ˆ£\ ¹(P)\,r(¹) + ¤\ i\ r(¹)^2\ ¡\ ¢(¹)^2\ ¢¤:$$

We thus infer that $d¹(P){=}d®$ has the same sign as $1\ ¡\ (®°^2{=}2)¹(P)(Var_P\ ')$.

7.7. Proof of Proposition 8

The equilibrium conditions (??) can be rewritten in polar coordinates (see the appendix for the derivation):

$$\begin{cases} \ln R = \ln R_0 + °\int_R^{+1}(1+½)\int_0^{2¼} g(½;µ;'^p;¢_1;¢_2)\,dµd½; \\ \int_R^{+1} p\overline{½}\int_0^{2¼}\cos(µ)\,g(½;µ;'^p;¢)\,dµd½ = 0; \\ \int_R^{+1} p\overline{½}\int_0^{2¼}\sin(µ)\,g(½;µ;'^p;¢)\,dµd½ = 0: \end{cases}$$

where $g(½;µ;'^p;¢) = \frac{¢_1¢_2 f\left(¢_1 p\overline{½}\cos(µ)+'^p_1;\ ¢_2 p\overline{½}\sin(µ)+'^p_2\right)}{2}$ and $¢_` = q\frac{2}{°®_`};\ ` = 1;2:$ First notice that, by symmetry of f around ¹, we have

$$g(½;µ;¹;¢) = g(½;µ+¼;¹;¢);\ µ\ 2\ f0;¼g$$

for any non-negative values of ½ and $¢_`;\ ` = 1;2$: Hence

$$\int_R^{+1} p\overline{½}\int_0^{2¼}\cos(µ)\,g(½;µ;¹;¢)\,dµd½ = 0;$$
$$\int_R^{+1} p\overline{½}\int_0^{2¼}\sin(µ)\,g(½;µ;¹;¢)\,dµd½ = 0;$$

are also satisfied for any non-negative values of R and $¢_`;\ ` = 1;2$: This implies that $'^p(¤)\ ´\ ¹$ for all ¤: The second part of the statement is a direct consequence of proposition 2.

7.8. Polar coordinates

Define:

$$\begin{aligned} x_\ell &= \frac{\varphi_\ell - \varphi_\ell^p}{\Delta_\ell}; \quad \ell = 1, 2. \\ P' &= \left\{ (x_1, x_2) : x_1^2 + x_2^2 \geq R \right\} \end{aligned}$$

where $\Delta_\ell = \sqrt{\frac{2}{\gamma \alpha_\ell}}$; $\ell = 1, 2$. System (??) then becomes[10]:

$$\begin{aligned} \ln R &= \ln R_0 + \gamma \left\{ \mu(P) + \int_{P'} \left(x_1^2 + x_2^2 \right) \Delta_1 \Delta_2 f\left(\Delta_1 x_1 + \varphi_1^p, \Delta_2 x_2 + \varphi_2^p \right) dx_1 dx_2 \right\} && (7.3) \\ \varphi_\ell^p &= \frac{1}{\mu(P)} \int_{P'} \left(\Delta_\ell x_\ell + \varphi_\ell^p \right) \Delta_1 \Delta_2 f\left(\Delta_1 x_1 + \varphi_1^p, \Delta_2 x_2 + \varphi_2^p \right) dx_1 dx_2; \quad \ell = 1, 2 && (7.4) \end{aligned}$$

Since

$$\int_{P'} \varphi_\ell^p f\left(\Delta_1 x_1 + \varphi_1^p, \Delta_2 x_2 + \varphi_2^p \right) dx_1 dx_2 = \varphi_\ell^p \mu(P);$$

we can rewrite (7.4) as:

$$\int_{P'} x_\ell f\left(\Delta_1 x_1 + \varphi_1^p, \Delta_2 x_2 + \varphi_2^p \right) dx_1 dx_2 = 0; \quad \ell = 1, 2. \tag{7.5}$$

Consider now the following transformation of variables to a polar coordinates system:

$$\begin{aligned} x_1 &= \sqrt{\rho} \cos(\theta); \\ x_2 &= \sqrt{\rho} \sin(\theta). \end{aligned}$$

The integral in (7.3) then becomes:

$$\int_R^{+\infty} \int_0^{2\pi} \rho \, g(\rho, \theta, \varphi^p, \Delta) \, d\theta \, d\rho;$$

---

[10] We are using:

$$\int_T f(x, y) \, dx dy = \int_S f(X(u, v), Y(u, v)) \, |J(u, v)| \, du dv$$

where $x = X(u, v)$; $y = Y(u, v)$, $J(u, v) = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix}$ and T is the image of S under the vector function $(X(\cdot, \cdot), Y(\cdot, \cdot))$.

and

$$\mu(P) = \int_R^{+\infty}\int_0^{2\pi} g(\rho;\theta;\varphi^p;\sigma)\, d\theta d\rho;$$

where we have defined $g(\rho;\theta;\varphi^p;\sigma) = \frac{\sigma_1\sigma_2 f\left(\sigma_1\sqrt{\rho}\cos(\theta)+\varphi_1^p;\sigma_2\sqrt{\rho}\sin(\theta)+\varphi_2^p\right)}{2}$. Similarly (7.5) can be rewritten:

$$\begin{aligned}\int_R^{+\infty}\sqrt{\rho}\int_0^{2\pi}\cos(\theta)\, g(\rho;\theta;\varphi^p;\sigma)\, d\theta d\rho &= 0;\\ \int_R^{+\infty}\sqrt{\rho}\int_0^{2\pi}\sin(\theta)\, g(\rho;\theta;\varphi^p;\sigma)\, d\theta d\rho &= 0.\end{aligned}$$

7.9. Equilibrium uniqueness in the Two Factor Model

We analyze a system of 5 equations in the 5 unkwowns $(\lambda_1;\lambda_2;\varphi_1^p;\varphi_2^p;R)$. The indifference conditions imply that $\lambda_i = \sqrt{2R/\alpha_i\gamma}$. Consider the functions

$$G(\varphi_1^p;\varphi_2^p;\lambda_1;\lambda_2) = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{bmatrix} \iint_{P(\varphi_1^p;\varphi_2^p;\lambda_1;\lambda_2)}(\varphi_1-\varphi_1^p)d\mu/\lambda_1^2 \\ \iint_{P(\varphi_1^p;\varphi_2^p;\lambda_1;\lambda_2)}(\varphi_2-\varphi_2^p)d\mu/\lambda_2^2 \end{bmatrix}$$

and

$$K(\varphi^p;\lambda) = -\frac{1}{2}\iint\left[\sum_{i=1}\left(\frac{\varphi_i-\varphi_i^p}{\lambda_i}\right)^2 - 1\right]1_{P(\varphi^p;\lambda)}d\mu.$$

We note that for a fixed φ, the function

$$k(\varphi;\varphi^p;\lambda) = \left[\sum_{i=1}\left(\frac{\varphi_i-\varphi_i^p}{\lambda_i}\right)^2 - 1\right]1_{P(\lambda)}$$

is convex in φ^p. Therefore $K(\varphi^p;\lambda)$ is strictly concave in φ^p. We observe that

$$\frac{\partial K}{\partial\varphi_i^p}(\varphi^p;\lambda) \equiv G_i(\varphi^p;\lambda).$$

The function $K(.;\lambda)$ thus reaches a maximum at unique point $\varphi^p(\lambda)$. The strict concavity of K implies that the matrix

$$\frac{\partial G}{\partial\varphi_p} = \frac{\partial^2 K}{\partial\varphi_p^2}$$

is negative definite. We note that

$$\frac{\partial K}{\partial \sigma_i}(\varphi^p;\sigma)=\frac{H_i}{\sigma_i^3}>0.$$

Since $\varphi^p(\sigma)$ is unique, there exists a unique solution set $P(\sigma)$, and we can define the functions

$$\begin{aligned} H_0(\sigma) &= \mu\,[P(\sigma)] \\ H_i(\sigma) &= \mu\,[P(\sigma)]\,Var_{P(\sigma)}(\varphi_i), \end{aligned}$$

for $i=1,2.$ In simulations, we observe that

- The mass H_0 is decreasing on R^2_+
- In general, the function H_0 is neither concave nor convex.

Since $H_0(\sigma) \le 1$ and $H_i(\sigma) \le Var(\varphi_i)$, we know that

$$\frac{\partial H_0}{\partial \sigma_j}<0 \qquad \text{and} \qquad \frac{\partial H_i}{\partial \sigma_j}<0$$

on a neighborhood of $\partial R^2_+.$

Consider the value function

$$V(\sigma)\equiv K[\varphi^p(\sigma);\sigma]=\frac{H_0(\sigma)}{2}-\frac{1}{2}\sum_{i=1}^{2}\frac{H_i(\sigma)}{\sigma_i^2}.$$

We know that V is defined on R^2_{++} and takes negative values. The envelope theorem implies that

$$\frac{\partial V}{\partial \sigma_j}=\frac{H_j}{\sigma_j^3}=\frac{1}{2}\frac{\partial H_0}{\partial \sigma_j}+\frac{H_j}{\sigma_j^3}-\frac{1}{2}\sum_{i=1}^{2}\frac{1}{\sigma_i^2}\frac{\partial H_i}{\partial \sigma_j}$$

and therefore

$$\sum_{i=1}^{2}\frac{1}{\sigma_i^2}\frac{\partial H_i(\sigma)}{\partial \sigma_j}=\frac{\partial H_0}{\partial \sigma_j}.$$

We infer that the value function V is increasing in its arguments. We note that when $\sigma \to \partial R^2_+$, we have $V(\sigma)\approx -\frac{1}{2}\sum_{i=1}^{2}\frac{Var(\varphi_i)}{\sigma_i^2} \to -\infty$. On the other hand, the value function V converges to zero when $\min \Lambda_i \to +\infty$. Since the Hessian matrix $D^2V = (\partial^2 V/\partial\Lambda_i\partial\Lambda_j)$ is symmetric, we infer that

$$\frac{\partial^2 V}{\partial\Lambda_i\partial\Lambda_j} = \frac{1}{\Lambda_i^3}\frac{\partial H_i}{\partial\Lambda_j} = \frac{1}{\Lambda_j^3}\frac{\partial H_j}{\partial\Lambda_i}$$

for all $i \neq j$.

Lemma. The value function $V(\Lambda)$ is strictly concave, and

$$\frac{\partial^2 V_i}{\partial\Lambda_i^2} \le -\frac{6H_i}{\Lambda_i^4}.$$

for all $i = 1, 2$.

Proof. Consider two vectors $\Lambda', \Lambda'' \in \mathbb{R}^L_{++}$, and let $\varphi' = \varphi^p(\Lambda')$, $\varphi'' = \varphi^p(\Lambda'')$. Let $t', t'' \in \mathbb{R}_+$, $t' + t'' = 1$. We consider $\Lambda = t'\Lambda' + t''\Lambda''$ and the vector $\varphi^* \in \mathbb{R}^L$ with coordinates

$$\varphi_i^* = \frac{\frac{t'}{\Lambda_i'^2}\varphi_i' + \frac{t''}{\Lambda_i''^2}\varphi_i''}{\frac{t'}{\Lambda_i'^2} + \frac{t''}{\Lambda_i''^2}} = \alpha_i'\varphi_i' + \alpha_i''\varphi_i''.$$

Since

$$(\varphi_i - \alpha_i'\varphi_i' - \alpha_i''\varphi_i'')^2 \le \alpha_i'(\varphi_i - \varphi_i')^2 + \alpha_i''(\varphi_i - \varphi_i'')^2,$$

we infer that

$$\begin{aligned}
\frac{(\varphi_i - \varphi_i^*)^2}{(t'\Lambda_i' + t''\Lambda_i'')^2} &= \left(\frac{t'}{\Lambda_i'^2} + \frac{t''}{\Lambda_i''^2}\right)(\varphi_i - \varphi_i^*)^2 - \\
&\quad \left(\frac{t'}{\Lambda_i'^2} + \frac{t''}{\Lambda_i''^2} - \frac{1}{(t'\Lambda_i' + t''\Lambda_i'')^2}\right)(\varphi_i - \varphi_i^*)^2 \\
&\le \frac{t'}{\Lambda_i'^2}(\varphi - \varphi_i')^2 + \frac{t''}{\Lambda_i''^2}(\varphi - \varphi_i'')^2 - \\
&\quad \left(\frac{t'}{\Lambda_i'^2} + \frac{t''}{\Lambda_i''^2} - \frac{1}{(t'\Lambda_i' + t''\Lambda_i'')^2}\right)(\varphi_i - \varphi_i^*)^2.
\end{aligned}$$

This implies

$$\begin{aligned}
\sum_{i=1} \frac{(\varphi_i - \varphi_i^*)^2}{\Lambda_i^2} - 1 &\le t'\left[\sum_{i=1} \frac{(\varphi - \varphi_i')^2}{\Lambda_i'^2} - 1\right] + t''\left[\sum_{i=1} \frac{(\varphi - \varphi_i'')^2}{\Lambda_i''^2} - 1\right] \\
&\quad - \sum_{i=1} \left(\frac{t'}{\Lambda_i'^2} + \frac{t''}{\Lambda_i''^2} - \frac{1}{\Lambda_i^2}\right)(\varphi_i - \varphi_i^*)^2
\end{aligned}$$

and therefore

$$\begin{aligned} k(\varphi;\varphi^*;\sigma) &\le \left\{ t' \left[\sum_{i=1}^L \frac{(\varphi_i-\varphi'_i)^2}{\sigma_i'^2} - 1 \right] + t'' \left[\sum_{i=1}^L \frac{(\varphi_i-\varphi''_i)^2}{\sigma_i''^2} - 1 \right] \right\} 1_{P(\varphi^*;\sigma)} \\ &\quad - \sum_{i=1}^L \left(\frac{t'}{\sigma_i'^2} + \frac{t''}{\sigma_i''^2} - \frac{1}{\sigma_i^2} \right) (\varphi_i-\varphi_i^*)^2 1_{P(\varphi^*;\sigma)} \\ &\le \left\{ t' \left[\sum_{i=1}^L \frac{(\varphi_i-\varphi'_i)^2}{\sigma_i'^2} - 1 \right] 1_{P(\varphi';\sigma')} + t'' \left[\sum_{i=1}^L \frac{(\varphi_i-\varphi''_i)^2}{\sigma_i''^2} - 1 \right] 1_{P(\varphi'';\sigma'')} \right\} 1_{P(\varphi^*;\sigma^*)} \\ &\quad - \sum_{i=1}^L \left(\frac{t'}{\sigma_i'^2} + \frac{t''}{\sigma_i''^2} - \frac{1}{\sigma_i^2} \right) (\varphi_i-\varphi_i^*)^2 1_{P(\varphi^*;\sigma)} \end{aligned}$$

We thus infer that

$$k(\varphi;\varphi^*;\sigma) \le t'k(\varphi;\varphi';\sigma')+t''k(\varphi;\varphi'';\sigma'') - \sum_{i=1}^L \left(\frac{t'}{\sigma_i'^2} + \frac{t''}{\sigma_i''^2} - \frac{1}{\sigma_i^2} \right) (\varphi_i-\varphi_i^*)^2 1_{P(\varphi^*;\sigma)}.$$

We integrate this relation and obtain

$$V(\sigma) \ge t'V(\sigma') + t''V(\sigma'') + \sum_{i=1}^L \left(\frac{t'}{\sigma_i'^2} + \frac{t''}{\sigma_i''^2} - \frac{1}{\sigma_i^2} \right) \int_{P(\varphi^*;\sigma)} (\varphi_i-\varphi_i^*)^2 d\mu;$$

since $V(\sigma) \ge K(\varphi;\varphi^*;\sigma)$. This relation holds for all σ, σ' and $\sigma'' \in R^L_{++}$. It implies that the value function V is strictly concave.

We now consider a fixed $\sigma \in R^L_{++}$, and the points $\sigma' = \sigma + \varepsilon d^{(i)}$, $\sigma'' = \sigma - \varepsilon d^{(i)}$. We see that

$$V(\sigma) \ge \frac{1}{2}[V(\sigma') + V(\sigma'')] + \left(\frac{1}{2\sigma_i'^2} + \frac{1}{2\sigma_i''^2} - \frac{1}{\sigma_i^2} \right) \int_{P(\varphi^*;\sigma)} (\varphi_i-\varphi_i^*)^2 d\mu$$

or equivalently

$$\begin{aligned} 0 \ge{}& \frac{1}{\varepsilon^2} \left\{ V[\sigma + \varepsilon d^{(i)}] + V[\sigma - \varepsilon d^{(i)}] - 2V(\sigma) \right\} + \\ & \frac{1}{\varepsilon^2} \left[\frac{1}{(\sigma_i+\varepsilon)^2} + \frac{1}{(\sigma_i+\varepsilon)^2} - \frac{2}{\sigma_i^2} \right] \int_{P(\varphi^*;\sigma)} (\varphi_i-\varphi_i^*)^2 d\mu. \end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$\frac{\partial^2 V}{\partial \sigma_i^2} + 6\frac{H_i}{\sigma_i^4} \le 0$$

which concludes the proof. ■

The matrix D^2V is therefore negative definite. Since

$$\frac{\partial^2 V}{\partial \sigma_i^2} = \frac{1}{\sigma_i^3}\frac{\partial H_i}{\partial \sigma_i} - \frac{3H_i}{\sigma_i^4} - 6\frac{H_i}{\sigma_i^4};$$

we infer that

$$\frac{\partial H_i}{\partial \sigma_i} - \frac{3H_i}{\sigma_i} < 0:$$

We now consider the function

$$u(\sigma;\alpha) \equiv \ln R_0 + \gamma H_0(\sigma) + \frac{\gamma^2}{2}\left[\alpha_1 H_1(\sigma) + \alpha_2 H_2(\sigma)\right]:$$

We can prove

Lemma. There exists a unique GEEP equilibrium.

Proof. Consider the functions $\sigma_i(R;\alpha) = \sqrt{2R/\alpha_i\gamma}$; and

$$v(R;\alpha) \equiv u[\sigma_i(R;\alpha);\alpha]:$$

Since $\alpha_i\gamma/2 = R/\sigma_i^2$ and

$$\frac{\partial \sigma_i}{\partial R} = \frac{\sigma_i}{2R};$$

we infer that

$$\begin{aligned}
\frac{\partial v}{\partial R} &= \gamma\left[\sum_{j=1}\frac{\partial H_0}{\partial \sigma_j}\frac{\sigma_j}{2R}\right] + \frac{\gamma^2}{2}\sum_{i=1}\alpha_i\sum_{j=1}\frac{\partial H_i}{\partial \sigma_j}\frac{\sigma_j}{2R}\\
&= \frac{\gamma}{2R}\left[\sum_{j=1}\sigma_j\frac{\partial H_0}{\partial \sigma_j}\right] + \frac{\gamma}{2}\sum_{i=1}\frac{1}{\sigma_i^2}\sum_{j=1}\frac{\partial H_i}{\partial \sigma_j}\sigma_j\\
&= \frac{\gamma(1+R)}{2R}\sum_{j=1}\sigma_j\frac{\partial H_0}{\partial \sigma_j};
\end{aligned}$$

implying

$$\frac{\partial v}{\partial R} = \frac{\gamma(1+R)}{R}\sum_{j=1}\sigma_i\sigma_j\frac{\partial^2 V}{\partial \sigma_i \partial \sigma_j} < 0:$$

The equilibrium equation

$$\ln R = v(R,\alpha)$$

has therefore a unique solution. $\blacksquare$

In order to establish the mononicity of the interest rate, we calculate

$$\frac{\partial v}{\partial \alpha_1} = \gamma \left(\frac{\partial H_0}{\partial \Lambda_1}\frac{\partial \Lambda_1}{\partial \alpha_1} + \frac{\gamma^2}{2} H_1(\Lambda) + \frac{\gamma^2}{2} \sum_{i=1} \alpha_i \frac{\partial H_i}{\partial \Lambda_1}\frac{\partial \Lambda_1}{\partial \alpha_1} \right).$$

Since $\partial \Lambda_1/\partial \alpha_1 = -\gamma/(2R\Lambda_1^3)$, we infer that

$$\begin{aligned} \frac{\partial v}{\partial \alpha_1} &= \frac{\gamma^2}{2} H_1(\Lambda) + \gamma \frac{\partial \Lambda_1}{\partial \alpha_1} \left[\frac{\partial H_0}{\partial \Lambda_1} + R \sum_{i=1} \frac{1}{\Lambda_i^2} \frac{\partial H_i}{\partial \Lambda_1} \right] \\ &= \frac{\gamma^2}{2} H_1(\Lambda) + \gamma (1+R) \frac{\partial \Lambda_1}{\partial \alpha_1} \frac{\partial H_0}{\partial \Lambda_1} \\ &= \frac{\gamma^2}{2} H_1(\Lambda) - \frac{\gamma^2}{2R\Lambda_1^3}(1+R) \frac{\partial H_0}{\partial \Lambda_1} \\ &= \frac{\gamma^2}{2} \left(H_1(\Lambda) - \frac{(1+R)}{R\Lambda_1^3} \frac{\partial H_0}{\partial \Lambda_1} \right). \end{aligned}$$

If we can show that $\partial H_0/\partial \Lambda_j < 0$ for all j, then we know that financial innovation increases the interest rate.

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