

Optimal Maturity of Government Debt with Incomplete Markets

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Abstract

In this paper we show how risk free bonds of different maturities can be used to replace state contingent debt in a general equilibrium dynamic optimal taxation problem. In particular, we show that if the state of the economy can only take a finite number N of values each period, then the government can support the complete markets Ramsey allocation issuing bonds of $J \leq N$ different maturities. We also show that the optimal maturity structure does depend on the relationship between the term structure of interest rates and government expenditures. In the case that interest rates are positively correlated with government expenditures in the Ramsey solution, then the government must hold short run assets and long term liabilities.

[†]PRELIMINARY AND INCOMPLETE (we do not have enough "maturity"!)

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1 Introduction

In this paper we show that in a dynamic economy where state contingent bonds cannot be issued, a Ramsey government may still implement the complete markets second best allocation by issuing risk-free bonds with a rich enough menu of different maturities.

The role for debt that pays in only some states of nature was emphasized by Lucas and Stokey (1983) as an instrument to smooth taxes across states. The active use of this financial instruments increases welfare and has been a key ingredient in most studies of optimal fiscal policy and optimal debt management since then¹. Lucas and Stokey also show that a proper management of the maturity structure of the debt makes the sequence of tax rates time consistent. In their model, however, if the government can fully commit to a tax policy, as we will assume in this paper, the maturity structure of the debt is irrelevant. In fact, this is the case for most general equilibrium models in the literature in which the government can fully commit to its future actions.

In a recent paper, Marcet, Sargent and Seppala (1999) solve a problem similar to the one in Lucas and Stokey but assume away the possibility of state contingent debt. Thus, in their model, the government issues only one period risk free bonds. They show that the stochastic properties of the optimal taxes and the debt are very different to the case in which the government can issue a full array of state contingent bonds². In particular, they show that in the economy with complete markets, optimal taxes inherit the autocorrelation properties from the stochastic process of government expenditures only, while in the economy with one period risk free bonds only, there is an additional state variable that introduces a unit root component on the solution of optimal taxes.

While Lucas and Stokey perform a purely normative exercise, MSS do attempt a positive exercise, since, they argue, their model without state contingent debt fits better the experience of XVIII century England, while XVIII century France seems to behave more like the model with at least some state contingent bonds achieved through default.

The purpose of this paper is to show that as two bonds of different maturity issued on the same date are different assets, they allow to span a greater set of allocations than a single maturity bond can. In fact, we show that if

¹See Chari, Christiano and Kehoe (1995) and the references therein.

²They also show, however, that the welfare effect of introducing state contingent bonds is very small.

the state of the economy has finite support, a rich enough set of maturities can complete markets. In particular, we show that in an infinite period model with uncertainty in which the shock can take on a finite number of values M each period, if the government can issue bonds with J different maturities, then if $J \geq M$; the government completes markets.

The intuition of the result is very simple. To complete markets, we require the same number of linearly independent assets as state of nature. Note that a j period bond today is different from a j period bond tomorrow, thus if the number of different assets at a particular date, conditional on a particular state, is the same as the number of shocks in that date for that particular state, and this is true for any date and state, the economy has the same number of assets as states.

The dependence of the term structure of interest rates on the state of the economy explains why markets can be completed with different maturities. A promise made at time t to deliver one unit of consumption at $t + 1$ is worth one unit of consumption in $t + 1$: However, a promise made at time t to deliver one unit of consumption at $t + 2$ is worth the same as a one period bond in $t + 1$; which does depend on the state of nature. Thus, if one is concerned about transferring wealth from t to $t + 1$; a one period bond is a risk free asset, while a two period bond is a risky asset.

In an economy with contingent assets, as in Lucas and Stokey, the government issues bonds that pay only when government expenditures are low. Thus, it can smooth taxes across states of nature. To understand how the maturity structure can replicate the result, recall that the price of a bond will in general depend on the government shock. If the Ramsey solution is such that private and public consumption are negatively correlated, then the interest rate will be relatively high when government expenditure is high, and the price of a bond will be lower. Thus, the way to insure against a positive shock in government expenditures is by holding a large amount of long term debt, whose value goes down when government expenditures go up. In fact, by explicitly solving a few examples, we show that the optimal maturity structure is characterized by holding short term assets and long term liabilities. In this way, the value of the outstanding debt will be negatively correlated with expenditures, as the Ramsey solution with complete markets indicate.

A qualification to the results must be made. In order to complete markets, the existing assets need to be linearly independent, i.e, the vector of returns of one and two period bonds must be different. We show that for this to be

the case, the term structure of interest rates in the Ramsey solution must be different across states, since it is the term structure that changes the prices of existing debt over time. However, we also show that as the term structure depends on the optimal tax rates, if there is enough assets, the government can always change slightly the taxes and approximate arbitrarily well the optimal state contingent allocation.

On the normative side, our paper shows that even though governments do not issue state contingent assets, a proper management of the debt maturity structure can do the job. On the positive side, our paper contributes to the literature on the properties of optimal debt and market structure. Since a wide range of maturities, as there is in real economies, approximates better than a single bond a full array of state contingent bonds, it may undo the quantitative importance of the martingale properties of the taxes obtained by MSS.

The paper is organized in the following way. Section 1 presents the model and shows the main result of the paper. Section 2 provides a series of examples to illustrate how the maturity of the debt should be managed. In section 3 we simulate a model with a maturity structure like the one existing in the US to evaluate if the stochastic properties of the optimal taxes resemble more the complete markets or the single one period risk free bond economies.

2 The Economy

The environment we analyze is very similar to the barter economy of Lucas and Stokey. This is a one perishable good economy with production. The representative agent is endowed with one unit of labor in every period. Labor is the single factor of production, and the technology is given by

$$c_t + x_t + g_t = 1; \quad t = 0; 1; 2; \dots; \text{ all } g_t \quad (1)$$

where x ; c and g represent leisure and private and public consumption respectively.

As it is standard in the Ramsey literature, government consumption is exogenous. We assume that it follows a stochastic process such that a single realization $g = (g_0; g_1; g_2; \dots)$ has the joint distribution F . We assume g_t to be the only source of uncertainty in this economy. We also assume that government expenditures can take one of a finite number N of values. This assumption allows us to make the point of the paper in a very clean way.

How the result can be extended to a continuum of values will be discussed in the end.

Preferences are represented by the following von Neumann-Morgenstern utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [u(c_t(g^t)) + v(x_t(g^t))] \quad (2a)$$

$\beta \in (0, 1)$; u and v are strictly increasing and strictly concave. The assumption of separability is in no way essential, as it will be obvious, but some of the results are cleaner.

The only tax available to the government is a τ_t rate tax levied against labor income $1 - \tau_t$. The government can also issue debt (that can be negative). We consider two environments, with different asset structures.

A Ramsey problem is to choose the allocation $\{c_t; x_t\}_{t=0}^{\infty}$ that maximizes (2a) subject to (1) and subject to the constraint that the allocation can be decentralized as a complete markets competitive equilibrium with income taxes. As it is standard in the literature, this last restriction can be represented by a single equation, the implementability constraint. This implementability constraint is obtained by replacing the complete markets competitive equilibrium prices in the life-time budget constraint of consumers³. In this case, the implementability constraint becomes

$$E_0 \sum_{t=0}^{\infty} \beta^t [c_t(g^t) \frac{U^0(c_t(g^t))}{U^0(c_0(g^0))} + (1 - \tau_t) x_t(g^t) \frac{V^0(x_t(g^t))}{U^0(c_0(g^0))}] = b_{i-1} \quad (3)$$

where b_{i-1} represents the outstanding liabilities inherited by the government⁴.

We will consider two full commitment environments with different asset markets. In the first one, we let the government issue one period state contingent debt. Let $b_t(g_{t+1}; g^t)$ stand for the contingent debt issued by the government in period t , given history g^t , to pay one unit of the consumption good in period $t+1$ contingent on the shock g_t being realized⁵. Alternatively, we let the government issue each period "risk free" bonds maturing at J dif-

³For details, see Lucas and Stokey (1983) or an appendix to this paper available from the authors upon request.

⁴Note that we are assuming that initial liabilities are all due on the first period. This is done just for simplicity, it is in no way essential to the results.

⁵Note that we do not consider the case of contingent debt with different maturities. For the case with commitment there is no loss of generality by doing this.

ferent dates. Let $b_t^j(g^t)$ denote the promise, made at t to deliver one unit of the good in period $t + j$ in any state of nature, after history g^t .

2.1 The solution with state contingent bonds

In this subsection we discuss the structure of the optimal debt when one-period state contingent bonds can be issued. This is the solution implemented by Lucas and Stokey.

The first sequence of constraints of the Ramsey problem (1) ensures aggregate material balance for every period and every contingency. The second constraint (3) ensures that the resulting allocation can be implemented as a competitive equilibrium with complete markets. To be able to represent all these equilibrium conditions on a single equation, the assumption of complete markets is essential. Note that we could write a similar condition at time t

$$z_{t+1}(g_t; g^{t+1}) = E_t \sum_{s=t}^{\infty} \beta^{s-t} \left[c_s(g^s) \frac{U^0(c_s(g^s))}{U^0(c_t(g^t))} + (1 - \alpha_s(g^s)) \frac{V^0(x_s(g^s))}{U^0(c_t(g^t))} \right] \quad (4)$$

The right hand side of this equation is the expected discounted value of the government surplus from period t (history g^t) on. That should be equal to the contingent liabilities the government has to honor on date t , given history g^t . In other words, for the government to be able to finance its deficits (surpluses) from date t on, conditioned on history g^t , a certain amount of wealth (debt) must be delivered at t , conditioned on that history, which we denote as $z(g_t; g^{t+1})$.

In a world with complete markets, we let $b_{t+1}(g_t; g^{t+1})$ represent outstanding debt obligations at time t ; given that state g^t has realized, conditional on history g^{t+1} ; so $b_{t+1}(g_t; g^{t+1}) = z_{t+1}(g_t; g^{t+1})$.

Note that given a Ramsey allocation, the right hand side of this equation is determined. The requirement of complete markets is equivalent to imposing no constraint linking the values of $b_{t+1}(g_t; g^{t+1})$ for alternative realizations of g_t ; except for (3), that imposes a present value condition for all $t; g^t$: On the other hand, if we assume, as in Marcat et. al. that there is only un-contingent one-period bonds, the restrictions that $b_{t+1}(g_t; g^{t+1}) = b_{t+1}(g^{t+1})$ for all $g_t; g^{t+1}$ and $t \geq 0$; must be added to the Ramsey problem. Thus, one way of understanding the Ramsey problem with complete markets is

letting the $b_{t_i-1}(g_t; g^{t_i-1})$ be unconstrained⁶. In fact, once the Ramsey problem with complete markets has been solved, one can disentangle the optimal one-period state contingent debt structure for every period and every state using equation (4).

2.2 A rich maturity structure of risk free bonds

In this section, we want to show how a structure of debt maturities maps into a structure of state contingent bonds if J is large enough. Let $b_{t_i-1}^{t_i+J}(g^{t_i-1})$ be debt issued at time t that is due at $t+J$; and let p_t^{t+i} be the price, in units of time t goods, of a promise to deliver one unit of the good at time $t+i$: Then, the time t value of outstanding obligations at time t ; given the realization g_t are given by

$$\sum_{i=0}^{J-1} p_t^{t+i}(g^t) b_{t_i-1}^{t_i+1}(g^{t_i-1}) + \dots + \sum_{i=0}^{J-1} p_t^{t+i}(g^t) b_{t_i-1}^{t_i+J}(g^{t_i-1}) + b_{t_i-1}^t(g^{t_i-1}) = \sum_{j=1}^J \sum_{i=0}^{j-1} p_t^{t+i} b_{t_i-1}^{t+i}$$

where the dependence of prices and quantities on the histories has been left implicit on the right hand side. The first sum of the left hand side represents the value of the debt issued at time $t+1$; the second one is the value of the debt issued at $t+2$; and so on. Note that of all debt obligations, only $b_{t_i-1}^{t_i+1}$ can depend on the history g^{t_i-1} : Thus, it is useful to write

$$\sum_{j=1}^J \sum_{i=0}^{j-1} p_t^{t+i} b_{t_i-1}^{t+i} = \sum_{i=0}^{J-1} p_t^{t+i}(g^{t_i-1}; g_t) b_{t_i-1}^{t+i}(g^{t_i-1}) + D_{t_i-1}(g^t)$$

Thus, while the amount of debt issued does not depend on the state of nature at t ; the value of the debt in terms of current consumption does, and therefore, the value of the debt obligations can be made contingent on the state of nature⁷. Note also, that while $D_{t_i-1}(g^t)$ does depend on the shock at t ; it cannot be affected by government decisions at $t+1$; since it is the value of the debt issued from period $t+J$ to $t+2$: Using equilibrium bond prices, the expression can also be written as

⁶Of course, (3) implies that the sequence of $b_{t_i-1}(g^t; g_{t_i-1})$ satisfies the relevant present value condition.

⁷Note that for this to be the case, it is essential to have debt of more than one period, since $p_t^t = 1$:

$$\sum_{j=1}^{\infty} \sum_{i=0}^{\infty} p_t^{t+i} b_{t,j}^{t+i} = \sum_{i=0}^{\infty} \frac{E_t[-^i u^0(c_{t+i}) | g^{t-1}; g^t]}{u^0(c_t)} b_{t,1}^{t+i}(g^{t-1}) + D_{t-1}(g^t)$$

where $E_t[-^i u^0(c_{t+i}) | g^{t-1}; g^t]$ represents the time t expected utility of $t + i$, conditioned on history $[g^{t-1}; g^t]$: This equation represents the value of government assets at time t after history $[g^{t-1}; g^t]$: We are interested in finding conditions such that, by appropriately choosing the values of $b_{t,j}^{t+i}(g^{t-1})$ for $j = 0; 1; 2; \dots; J-1$; the government can reproduce the required Ramsey values $z_{t-1}(g^{t-1}; g^t)$ for all $t; (g^{t-1}; g^t)$: Thus, it has to be the case that

$$\sum_{j=1}^{\infty} \sum_{i=0}^{\infty} p_t^{t+i}(g^t) b_{t,j}^{t+i}(g^{t-1}) = z(g_t; g^{t-1})$$

or

$$z(g_t; g^{t-1}) = \sum_{i=0}^{\infty} \frac{E_t[u^0(c_{t+i}) | g^{t-1}; g^t]}{u^0(c_t)} b_{t,1}^{t+i}(g^{t-1}) + D_{t-1}(g^t)$$

for all $g_t; g^{t-1}$; and $t \geq 0$:

Let the vector of government debt of different maturities issued at time $t-1$ be

$$b_{t-1}(g^{t-1}) = \begin{pmatrix} b_{t-1}^1(g^{t-1}) \\ b_{t-1}^2(g^{t-1}) \\ \vdots \\ b_{t-1}^J(g^{t-1}) \end{pmatrix};$$

the matrix of returns of the debt in period t ,

$$A_t(g^t) = \begin{pmatrix} 1 - \frac{E_t[u^0(c_{t+1}) | g^{t-1}; g_1]}{u^0(c_t)} & \dots & -J_{i-1} \frac{E_t[u^0(c_{t+J_i-1}) | g^{t-1}; g_1]}{u^0(c_t)} \\ 1 - \frac{E_t[u^0(c_{t+1}) | g^{t-1}; g_2]}{u^0(c_t)} & \dots & -J_{i-1} \frac{E_t[u^0(c_{t+J_i-1}) | g^{t-1}; g_2]}{u^0(c_t)} \\ \vdots & \dots & \vdots \\ 1 - \frac{E_t[u^0(c_{t+1}) | g^{t-1}; g_N]}{u^0(c_t)} & \dots & -J_{i-1} \frac{E_t[u^0(c_{t+J_i-1}) | g^{t-1}; g_N]}{u^0(c_t)} \end{pmatrix};$$

and the vector of wealth transfers

$$Z_t(g^t) = \begin{pmatrix} Z_t(g^{t-1}; g_1) \\ Z_t(g^{t-1}; g_2) \\ \vdots \\ Z_t(g^{t-1}; g_N) \end{pmatrix};$$

Then, the system of equations for period t ; given history g^{t-1} can be written as,

$$A_t(g^t)b_{t-1}(g^{t-1}) + D_{t-1}(g^t) = Z_t(g^t)$$

If the matrix of returns $A_t(g^t)$ is nonsingular we obtain

$$b_{t-1}(g^{t-1}) = A_t(g^t)^{-1}[Z_t(g^t) - D_{t-1}(g^t)]$$

which means that we can always find an intertemporal strategy of debt issues to support the Ramsey allocation.

A necessary condition for the Ramsey to be implementable is that the government can issue risk free bonds in at least as many maturities as possible realizations of the government expenditure, i.e., $J \geq N$: A sufficient condition is that the yield curve differs across states of nature.

In the Ramsey allocation, the interest rate between period t and $t + j$ depends both on the marginal utility in each state, and on the conditional probability of each state. Thus, the sufficient condition can be expressed as conditions on the vector of marginal utilities and conditions on the transition probability matrix. Indeed, assume that the expenditure shock follows a Markov chain with transition matrix π :

As we argued before a Ramsey allocation can be supported by risk free bonds of different maturities if the matrix $A_t(g^t)$ is nonsingular. This matrix can be written as

$$A_t(g^t) = \begin{pmatrix} 0 & U^0(c(g_1)) & 0 & \dots & 0 & 1 \\ 0 & 0 & U^0(c(g_2)) & \dots & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \vdots & \vdots & \dots & U^0(c(g_N)) & \vdots \end{pmatrix} \begin{matrix} B \\ A \end{matrix}$$

where

$$B = \begin{pmatrix} 0 & U^0(c_1) & -E[U^0(c_{g_{t+1}}) | g_t = g_1] & \dots & -j_i^{-1} E[U^0(c_{g_{t+j_i-1}}) | g_t = g_1] & 1 \\ U^0(c_2) & -E[U^0(c_{g_{t+1}}) | g_t = g_2] & \dots & -j_i^{-1} E[U^0(c_{g_{t+j_i-1}}) | g_t = g_2] & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ U^0(c_N) & -E[U^0(c_{g_{t+1}}) | g_t = g_N] & \dots & -j_i^{-1} E[U^0(c_{g_{t+j_i-1}}) | g_t = g_N] & \vdots \end{pmatrix} \begin{matrix} A \\ \vdots \end{matrix}$$

Thus, as $\text{rank}(AB) = \min(\text{rank}(A); \text{rank}(B))$ and the diagonal matrix with nonzero elements of the diagonal is nonsingular, then $\text{rank}(AB) = \text{rank}(B)$:

Using standard properties of Markov chains and letting the vector of marginal utilities for the N different states be

$$U_C = \begin{pmatrix} U^0(c(g_1)) \\ U^0(c(g_2)) \\ \vdots \\ U^0(c(g_N)) \end{pmatrix}$$

the j th column of the matrix B can be written as

$$\beta^{j-1} \beta^j U_C$$

Then, the Ramsey allocation can be supported with bonds of N different maturities if the vectors $\beta^{j-1} \beta^j U_C$; $j = 1, \dots, N$ are linearly independent. It is interesting to discuss the roles that β and U_C play.

First, note that if U_C is a vector of constants, so the interest rate does not depend on the state of nature, then all vectors in matrix B are linearly dependent, such that the rank is equal to one. In this case, the term structure of interest rates is flat for every state of nature⁸. Thus, bonds of different maturities are the same asset so the economy behaves as in Marcet, Sargent and Seppala. Second, note that if $N > 2$; and the shocks are iid, then $\beta^j = \beta$ for all j : Thus, if there are more than two states, it is required that the shocks have some persistence for bonds of different maturity but longer than two periods to be different assets. If the shocks are iid, bonds of more than two periods are equivalent⁹.

In summary, if consumption is sensitive to shocks, then U_C and βU_C are linearly independent. If, in addition, the shocks have some persistence, then $\beta^j U_C$ and $\beta^i U_C$ will also be linearly independent for all $i, j > 0$: Note also that these two assumptions imply that $\beta^j U_C$ and U_C are linearly independent too for all $j > 1$, since the operator β^j is a contraction for $j > 1$ ¹⁰.

PACO: Esto quedo bastante bien. Las dos condiciones necesarias y suficientes son que el vector U_C no sea un vector de constantes y que la matriz β

⁸A utility function in which labor and consumption are separable and labor enters linearly would deliver this result.

⁹In fact, if shocks were correlated for N periods only, bonds with maturities more than N periods would be equivalent.

¹⁰Note that the operator β^j ; when applied to the vector U_C reduces the maximum value among the components of the vector, and increases the minimum value. Thus, since the linear operator is monotonic, $\beta^j U_C$ and U_C ought to be linearly independent.

no sea iid. Con estas dos condiciones estamos hechos. Lo que queda por investigar, y no creo que podamos hacer nada teóricamente, por lo menos antes de fines de abril, es como la madurez de la deuda depende de dos cosas.

1. Si el consumo baja cuando el gasto sube, como antes.

2. La persistencia del shock. Si puedes, prueba numericamente con 3 estados y anda haciendo variar el sigma del ocio, que te va a afectar la slope del consumo y haciendo variar el nivel de persistencia, empezando por positiva y chiquita y aumentandola y por negativa y chiquita (en valor absoluto) y aumentandola. Supongo que lo que debería pasar es que cuanto mas slope el consumo como lo tenemos en el paper y mas positiva la persistencia, va a ser short en short y long en long, y ademas los montos absolutos van a ir decreciendo. Por el otro lado, si el slope es al revés de como lo tenemos en el paper y la persistencia es negativa, la cosa va a ir al revés. Seris bueno jugar numericamente con eso, pero no lo hagas todavía que estamos mirando datos de USA para calibrar un poco mejor aunque sea los niveles de los shocks y la persistencia.

3 Some Examples

In this section we illustrate the structure of maturities associated with the Ramsey allocation for several examples, to give a flavor of what the optimal debt maturity structure ought to be. First, we show it using 3 examples from Lucas and Stokey plus an additional one with three states of nature.

Before doing so, it is worth emphasizing a few properties of the Ramsey solution¹¹. If the Ramsey solution is interior, it has to satisfy the following first order conditions

$$(1 + \lambda_t)U^0(c_t(g^t)) + \lambda_t c_t(g^t)U^{00}(c_t(g^t)) = \lambda_{t+1}(g^{t+1}); \quad t = 0; 1; 2; \dots; \text{all } g^t:$$

$$(1 + \lambda_t)V^0(x_t(g^t)) + \lambda_t (x_t(g^t) - 1)V^{00}(x_t(g^t)) = \lambda_{t+1}(g^{t+1}); \quad t = 0; 1; 2; \dots; \text{all } g^t:$$

Using these first order conditions together with equation (1) it is possible to solve for λ_t ; c_t ; x_t as a function of the multiplier λ_t and the value of the shock g_t only, so we can write them as $\lambda_t(g_t; \lambda_t)$; $c_t(g_t; \lambda_t)$; $x_t(g_t; \lambda_t)$: This means that, as the multiplier is the same for every period and history, every time that

¹¹For details, see Lucas and Stokey (1983).

government expenditures takes the same value, the allocation $c_t; x_t$ is the same.

As it may be clear already from last section, a key element to determine the optimal debt structure is the behavior of the yield curve associated with the Ramsey allocation. This will be given by the holding return of the different debts, which in turn depends on the correlation between consumption and government expenditures.

Lemma 1 : Given an interior solution, the value of consumption will be relatively low when g is relatively high if $V''(x) > 0$:

Proof: Using the necessary conditions for an optimum and the resource constraint we get an equation relating c_t and g_t :

$$(1 + \tau)U'(c_t) + \tau c_t U''(c_t) = (1 + \tau)V'(1 - c_t - g_t) - \tau(c_t + g_t)V''(1 - c_t - g_t)$$

applying the implicit function theorem to get an expression for $\frac{\partial c_t}{\partial g_t}$:

$$\frac{\partial c_t}{\partial g_t} = \frac{(1 + 2\tau)V''(\cdot) - \tau(c_t + g_t)V'''(\cdot)}{(1 + 2\tau)U''(\cdot) + \tau c_t U'''(c_t) + (1 + 2\tau)V''(\cdot) - \tau(c_t + g_t)V'''(\cdot)}$$

>From the second order conditions we know that the denominator is negative. As $V''(\cdot) > 0$ and $\tau > 0$ ¹², the numerator is also negative.

NOTE: Paco, este resultado tiene que estar relacionado, como buen Ramsey problem que es, a lo que le pasa a la elasticidad del consumo cuando sube g , si el impuesto es mas alto, la elasticidad es mas alta, this better be true.

Example 2 :(corresponds to example 5 in LS) Let $b_0^j = 0$; $j = 1; 2$; let $g_t = 0$ for all $t \in T$ and let $g_T = G > 0$ with probability θ and $g_T = 0$ with probability $1 - \theta$:

As we argued above, the first order conditions of the Ramsey problem imply that the allocation will be the same every period and state where g_t is the same. Thus, using the implementability constraint (3)

¹²See Lucas and Stokey equation 3.3, or the appendix to this paper, available from the authors upon request.

$$\sum_{t=0}^T \beta^t [cU^0 + (x_i - 1)V^0] + \beta^{T-1} f[cU^0 + (x_i - 1)V^0] - [eG^0 + (x_i - 1)V^0]g = 0$$

where $c = c_t(g_t = 0)$; $e = c_t(g_t = G)$; and the same interpretation holds for all other variables. If we let $\lambda = cU^0 + (x_i - 1)V^0$; we can write the equation above as

$$\sum_{t=0}^T \beta^t \lambda + \beta^{T-1} f[\lambda] - [e]g = 0 \quad (5)$$

Note also that if we multiply the first order conditions by the corresponding variable for the periods and states where the government expenditure is zero, we obtain

$$(1 + \rho)cU^0 + \rho c^2U^{00} = c^1;$$

$$(1 + \rho)(x_i - 1)V^0 + \rho(x_i - 1)^2V^{00} = (x_i - 1)^1;$$

and adding the two

$$(1 + \rho)[cU^0 + (x_i - 1)V^0] + \rho[c^2U^{00} + (x_i - 1)^2V^{00}] = [c + x_i - 1]^1 = 0$$

or

$$(1 + \rho)\lambda + \rho[c^2U^{00} + (x_i - 1)^2V^{00}] = [c + x_i - 1]^1 = 0$$

But $\rho > 0$ and both U and V are concave, so $\lambda > 0$: Thus, it follows from (5) that $g < 0$:

With one-period state contingent debt, the optimal debt structure has to satisfy, for $t = 0; 1; \dots; T - 1$

$$\lambda_i (1 + r_i)^{-t} \frac{\lambda_i}{1 + r_i} = \beta^t U^0 b_{t+1}$$

Thus, for the first periods, the government runs surpluses and accumulates assets. Note that as there is no uncertainty, only risk free assets are issued. For $t > T$; the debt satisfies

$$\frac{\lambda_i}{1 + r_i} = U^0 b_{t+1}$$

since there is no uncertainty from there on. Note that this must also hold for period T if $g_T = 0$; since all periods are alike from there on. Finally, at period T ; the debt must satisfy

$$! + \textcircled{b}_{T_i-1}^{-\Theta^0} + (1_i \textcircled{b})b_{T_i-1}^{-U^0} = b_{T_i-2}U^0$$

and recall that b_{T_i-2} is negative and $!$ positive. Also b_{T_i-1} is positive, so \textcircled{b}_{T_i-1} is negative.

Thus, the government will be running budget surpluses, accumulating assets, up to date T . At date $T_i - 1$ the government will be selling bonds that pay on the event that the expenditure is low, and buying bonds that pay contingent on the expenditure being high.

We now show that it is possible to replicate the same allocation with one and two period risk free bonds. The implementability constraints, evaluated at T in both states, must satisfy

$$\begin{aligned} (b_{T_i-2}^T + b_{T_i-1}^T)\Theta^0 + -U^0 b_{T_i-1}^{T+1} &= \textcircled{b} + \frac{-}{1_i}! \\ (b_{T_i-2}^T + b_{T_i-1}^T)U^0 + -U^0 b_{T_i-1}^{T+1} &= ! + \frac{-}{1_i}! \end{aligned}$$

As there is no uncertainty until period T ; we assume that the government issues only one period debt (actually assets!) for $t < T_i - 1$; so $b_{T_i-2}^T = 0$: We can write the system as

$$\begin{aligned} \tilde{A} \begin{matrix} \Theta^0 & -U^0 \\ U^0 & -U^0 \end{matrix} \begin{matrix} ! \\ \tilde{A} \end{matrix} \begin{matrix} b_{T_i-1}^1 \\ b_{T_i-1}^2 \end{matrix} &= \begin{matrix} \tilde{A} \\ \tilde{A} \end{matrix} \begin{matrix} \textcircled{b} + \frac{-}{1_i}! \\ ! + \frac{-}{1_i}! \end{matrix} \end{aligned}$$

so the solution for the values of debt issued at $T_i - 1$ is

$$\begin{aligned} b_{T_i-1}^1 &= \frac{-U^0(\textcircled{b}_i !)}{-U^0(\Theta^0_i U^0)} \\ b_{T_i-1}^2 &= \frac{\frac{1}{1_i}(\Theta^0_i [(1_i^-)\textcircled{b} + -!]U^0)}{-U^0(\Theta^0_i U^0)} \end{aligned}$$

Recall that $! > 0$ and $\textcircled{b} < 0$: By Lemma 1, consumption is lower when government expenditures are higher, so $\Theta^0 > U^0$: Thus, the government accumulates assets in the short run and issues debt in the long run.

With one and two period bonds, we can replicate the payoff of the contingent debt, provide that the conditions of the Theorem 1 are satisfied, being long on the debt worth more on the state with high expenditure, and short on the other debt. The relevant price to determine the optimal maturity structure of government debt is the interest rate of period T , since this will determine the price of the one period debt of period T , that will be the price at period T of the two period debt issued on period $T-1$.

Lemma 1 established that consumption is relatively low when the expenditure is relatively high, implying that the interest rate is higher on this state (i.e., the price of the two period bonds that are being carried from the previous period is low). Then, the government at $t = T-1$ will be issuing long term debt and accumulating short term assets.

Given that the interest rate at T does depend on the value of g_T , the matrix of returns is nonsingular and the system has a (unique) solution. Note however, that the matrix of returns can be singular. If V is linear in leisure, then the value of consumption¹³ is constant across states and time periods, so $U^0 = U^1$: If this is the case, the government can approximate arbitrarily well the Ramsey solution by letting the income tax at time T if $g_T = G$ be equal to the Ramsey tax plus epsilon, and reduce accordingly the tax in all other states and periods such that the budget constraint holds. Note that for all positive values of epsilon the matrix of returns will be nonsingular. As epsilon goes to zero, welfare goes to Ramsey.

Example 3 : (corresponds to example 7 in LS) Let $b_0^j = 0 \ j=1,2$; let $g_0 = G > 0$: If $g_t = G$, then $g_{t+1} = G$ with probability θ and $g_{t+1} = 0$ with probability $1-\theta$: If $g_t = 0$; then $g_{t+1} = 0$:

Again, there will be only two possible taxes in equilibrium and, correspondingly, two possible allocations. Let τ and τ' have the same interpretations as before. The implementability constraint at time zero can be written as

$$E_0 \sum_{t=0}^{\infty} \beta^t \tau_t = 0$$

or

$$\frac{\tau}{1+i} + \frac{\tau'(1+i\theta)}{(1+i)(1+i\theta)} = 0$$

¹³This trivially follows from the first order conditions of the Ramsey problem.

As before, $\theta > 0$ and $\epsilon < 0$: Now, at $t > 0$; it has to be the case that when expenditures are high,

$$b_{t+1}(G)\theta^0 = \frac{\epsilon}{1+i} + \frac{\theta - (1+i)\theta^0}{(1+i)(1+i)^0} = 0$$

while

$$b_{t+1}(0)\theta^0 = \frac{\theta}{1+i}$$

if expenditures are low. Thus, if expenditures are high, the government runs a deficit and finances it with debt that only pays if next period expenditures are low. While expenditures are high, the government repeats the policy. If at some t expenditures are low, the government will be running a surplus forever, to pay the interest on the debt and will roll over the debt.

Similarly to the previous example, the government will be running budget surpluses when expenditures are zero and deficits when they are positive. Initial net liabilities will be zero if expenditures are positive and positive in case they become zero. Thus, the government issues debt that pays only if expenditures become zero.

Also as in the previous example, the value of the two period debt is low if g is high, so the same intuition applies, and the optimal debt structure is similar: long term liabilities and short term assets holdings.

We show now how the same allocation can be implemented by appropriately issuing one and two period bonds. In this case, the implementability constraints at time one, in each state, must satisfy

$$\begin{aligned} b_0^1 \theta^0 + -EU_2^0 b_0^2 &= \frac{\epsilon}{1+i} + \frac{\theta - (1+i)\theta^0}{(1+i)(1+i)^0} \\ b_0^1 U^0 + -EU_2^0 b_0^2 &= \frac{\theta}{1+i} \end{aligned}$$

or

$$\begin{aligned} \bar{A} \begin{pmatrix} b_0^1 \\ b_0^2 \end{pmatrix} &= \begin{pmatrix} \theta^0 - EU_2^0 \\ U^0 - EU_2^0 \end{pmatrix} + \bar{A} \begin{pmatrix} \epsilon + \frac{\theta - (1+i)\theta^0}{1+i} \\ \theta + \frac{\theta}{1+i} \end{pmatrix} \end{aligned}$$

and the solution will be

$$b_0^1 = \frac{1}{1 - i^*} \frac{b + i}{\theta^0 + U^0}$$

$$b_0^2 = \frac{1}{-(\theta^0 + U^0)(\theta^0 + (1 - i^*)U^0)} \left[\frac{\theta^0}{1 - i^*} + \frac{U^0}{1 - i^*} \left(\frac{b(1 - i^*)}{1 - i^*} + \frac{i^*(1 - i^*)}{1 - i^*} \right) \right]$$

As before, $b < 0$; $i^* > 0$ and $\theta^0 + U^0 > 0$. Also, $\left(\frac{b(1 - i^*)}{1 - i^*} + \frac{i^*(1 - i^*)}{1 - i^*}\right) < i^*$; since it is a weighted sum of b and i^* : Thus, the optimal policy implies to hold short term assets and long term liabilities.

How about the optimal debt policy for general t ? Note that the solution is not stationary, since at the beginning of time the government does not inherit a two period bond, while this will be the case from period one on. Thus, the optimal debt structure at time $t \geq 2$ while government expenditure is high, must satisfy

$$(b_{t-2}^t + b_{t-1}^t)\theta^0 + -E_t U_{t+1}^0 b_{t-1}^{t+1} = \frac{b}{1 - i^*} + \frac{i^*(1 - i^*)}{(1 - i^*)(1 - i^*)}$$

$$(b_{t-2}^t + b_{t-1}^t)U^0 + -E_t U_{t+1}^0 b_{t-1}^{t+1} = \frac{i^*}{1 - i^*}$$

The only difference with period 1 is that $b_{t-2}^t \neq 0$. The solution now will be

$$\begin{matrix} \bar{A} & i^* & \bar{A} & \theta^0 & -EU^0 & i^* & 0 & \frac{b}{1 - i^*} & + & \frac{i^*(1 - i^*)}{(1 - i^*)(1 - i^*)} & \bar{A} & b_0^2 \theta^0 & \\ b_{t-1}^t & = & U^0 & -EU^0 & & & & \frac{i^*}{1 - i^*} & & & A_i & b_0^2 U^0 & g \end{matrix}$$

Thus, the amount of long term assets issued is the same as before, while the amount of short term debt issued is larger, given by

$$b_{t-1}^t = b_0^1 + b_{t-2}^t$$

The intuition is simple. Since what matters is the net value of assets due at t , regardless of when they were issued, the accumulation of assets after period one has to compensate for the two period debt issued two periods before.

NOTE: There is something that bothers me. I do not understand the cash flows. If tax receipts are the same, and long term debt is the same, how come short term debt is not, which is the third term in the equation? Am I missing the effect of the price on the value of the debt? dontinso.

Example 4 : Let $b_0^j = 0 \quad j=1,2$; $g_0 = g^h$ and assume the government expenditures follows a Markov process with two states, g^h and 0 , where $g^h > 0$; with transition matrix $P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$:

The implementability constraint in period zero implies that

$$E_0 \sum_{t=0}^{\infty} \beta^t g_t = 0 = E_0 \sum_{t=0}^{\infty} \beta^t g_t = g^h = E_h$$

and this will also be the case for every t such that $g_t = g^h$; given the recursive structure of the economy. Similarly

$$k = E_0 \sum_{t=0}^{\infty} \beta^t g_t = 0 = E_l$$

where k is constant to be determined. Given the transition matrix, and letting l and h be the value of the surplus when government expenditures are low and high respectively, as before, E_h ; E_l must satisfy

$$\begin{aligned} 0 &= E_h = l + \beta [\frac{1}{4} E_h + \frac{1}{4} E_l] \\ k &= E_l = l + \beta [\frac{1}{4} E_h + \frac{1}{4} E_l] \end{aligned}$$

or

$$\begin{aligned} 0 &= E_h = l + \beta \frac{1}{4} k \\ k &= E_l = l + \beta \frac{1}{4} k \end{aligned}$$

which means that $k > 0$: Thus, the optimal maturity structure the first period will satisfy

$$\begin{aligned} b_0^1 \theta^0 + \beta E U_2^0 b_0^2 &= 0 \\ b_0^1 U^0 + \beta E U_2^0 b_0^2 &= k \end{aligned}$$

which can be written

$$\begin{bmatrix} b_0^1 \\ b_0^2 \end{bmatrix} = \begin{bmatrix} \theta^0 & -E U_2^0 \\ U^0 & -E U_2^0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ k \end{bmatrix}$$

and the result will be

$$b_0^1 = \frac{i k}{\Theta^0 - U^0}$$

$$b_0^2 = \frac{\Theta^0}{-E(U_2^0)} \left[\frac{i k}{\Theta^0 - U^0} \right]$$

Again, the optimal structure involves holding short run assets and long run liabilities. The optimal structure for $t > 1$ satisfies

$$(b_{t-2}^t + b_{t-1}^t)\Theta^0 + -EU_{t+1}^0 b_{t-1}^{t+1} = 0$$

$$(b_{t-2}^t + b_{t-1}^t)U^0 + -EU_{t+1}^0 b_{t-1}^{t+1} = k$$

and the solution will be as before,

$$b_{t-1}^t = b_0^1 - b_{t-2}^t$$

$$b_{t-1}^{t+1} = b_0^2$$

Discussion of the results In all the examples discussed, optimality calls for accumulating short run asset and holding long run liabilities. Lemma 1 is behind this result. Note that in the three cases, in the solution of b_{t-1}^t ; i.e., the optimal amount of one period debt to issue, the term $(\Theta^0 - U^0)$ is in the denominator. Lemma 1 establishes the conditions under which that term is larger than zero. Recall that Θ^0 is the marginal utility of consumption when government expenditures are high, which, if the conditions of Lemma 1 are satisfied, is larger than U^0 . The reason for this is that optimality calls for a larger income tax when government expenditures are higher. Being the solution of a Ramsey problem, it has to be the case that the demand elasticity of consumption is higher when government consumption is higher under the conditions of Lemma 1.

Recall that tax smoothing across states with contingent debt is achieved by issuing debt that does not pay off when government expenditures are high. To reproduce the result with short and long term debt, and if the interest rate goes up when government expenditures go up, it is optimal to have a portfolio whose value is increasing with the current interest rate, i.e., a portfolio with short run assets and long run debt.

Example 5 Let $b_0^j = 0; j = 1; 2; 3; g_0 = g^h$ and assume the government expenditures follows a markov process with three states, $g^h; g^m$ and g^l , where

$$g^h > g^m > g^l; \text{with transition matrix } P = \begin{matrix} & \begin{matrix} h & m & l \end{matrix} \\ \begin{matrix} h \\ m \\ l \end{matrix} & \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \end{matrix}$$

NOTA: Este no lo termine, ya que no pude demostrar que los valores del superavit, τ , son decrecientes con el valor del gasto. En los casos anteriores esto funciono porque supusimos que el gasto era cero en el estado bajo y habia solo dos estados. Pero hay que chequearlo.

4 Simulations. IN PROGRESS

As we mentioned before, Marcet et al have shown that the stochastic properties of optimal policies depend crucially on the market structure. When markets are complete, as in Lucas and Stokey, the serial correlation of optimal taxes and government debt are tied closely to that one of government expenditures. On the other hand, when the only asset is a one-period risk free bond, a series of constraints must be imposing, that ensure that $z(g_t; g^{t-1}) = z(g^{t-1})$ for all $t; g^{t-1}$. They show that the multiplier of this constraint imparts a unit-root component to the solution of the optimal tax rate.

As we showed in this paper, the existence of several maturities allows the government to at least partially complete markets. In a discrete time model like the one we have it would be impossible to complete markets in the environment computed by Marcet et al, since the support of the shock is a continuum. Thus, allowing for the maturities that the US government issues will complete only partially markets.

The numerical question we address in this section is if a model like the one in Marcet et al implies that optimal taxes behave as random walks when the government uses optimally the maturity structure to better approximate the Ramsey allocation.

TO BE DONE

NOTE: we also have to work out the first order conditions and show that, hopefully, the more maturities available, the less random walk components in the first order conditions that determine the optimal tax rates.

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