Exogeneity in VAR-ECM models with purely exogenous long-run paths with an illustration to Mexico real exchange rate determination[1]

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I. Introduction

Vector error correction models (VAR-ECM) have now become an essential part of time series econometricians toolkit, and are familiar to many macroeconomists as the framework within which such techniques as weak exogeneity, strong exogeneity for instance may be analysed. A detailed treatment of these concepts has already been provided by the important literature devoted to these topics which has considerably developed over the last decade, notable contribution being (Johansen (1992a, 1992b), (1995),Urbain (1992), (1995), Boswijk (1991), (1992), (1995), Hendry and Mizon (1993),…). A common feature of these papers is that most of them suppose implicitly that the applied econometrician has a potential interest in all cointegrating relations existing between the variables being investigated. However it must be pointed out that really very often in empirical applications, the applied econometrician is chiefly interested in a subset of equilibrium relations, namely those linking both exogenous and endogenous variables which belong to the equations describing the evolution of the endogenous variables (conditional model) and less frequently in all cointegrating relations, for instance by those involving only exogenous variables. Furthermore a typical difficulty sometimes arises for macroeconomists when cointegration tests suggest in empirical applications the existence of r cointegrating vectors, whereas according to economic theory there should only exist m equilibrium relations with m < r. These two examples clearly

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illustrate the idea that considering all long run relations as potential interest for applied economists may be in some case too restrictive[3].

Moreover even theoretically an issue arises since these exogeneity conditions constrained the equations describing the evolution of the exogenous variables (marginal model) to be a VAR in difference, whereas one could have easily faced the possibility that these equations contain purely exogenous long run paths that do not belong to the conditional model. Therefore given those observations, it may be useful if less restrictive exogeneity conditions were available.

In this purpose we propose in this paper necessary and sufficient weak and strong exogeneity conditions[4], which allow the existence of purely exogenous long run paths in the marginal model as well as in the conditional model and that express themselves as minimum conditions on the parameters of a canonical representation, in which the nullity of some parameter blocks imply no loss of generality. Furthermore these conditions offer investigators the possibility to be only interested in a subset of cointegrated vectors.

This paper is organised as follows : section II sets out the general VAR-ECM framework. Section III introduces the canonical representation of the long run matrix Π in the I(1) case and proposes less restrictive exogeneity conditions which can easily be implemented in empirical applications, while section IV presents a more general canonical representation of the long run matrix Π, which can give rise to I(2) behaviour of the endogenous variables and reconsiders the previous exogeneity conditions in this framework. Section V deals with inference and testing which are conducted within the setting proposed by Johansen, whereas section VI provides an empirical application using quarterly data for Mexico real exchange rate and illustrates the use of these new exogeneity conditions. Finally, concluding remarks are presented in section VII.

II. Cointegrated vector autoregressions

[3] A similar remark has also been made by Ericsson (1995) in his careful discussion of Boswijk's paper (1995) on structural ECMs, concerning the weak exogeneity hypothesis for all cointegrating vectors (assumption (ii)) adopted by Boswijk : Ericsson asserts that this is an "overly strong hypothesis", since according to him, "any individual empirical investigation might reasonably restrict its focus to only a subset of the cointegrating vectors in the economy". Furthermore he provides analytical and empirical examples in which this hypothesis is violated.

[4] In this paper, we shall confine ourselves to the concepts of weak and strong exogeneity proposed by Richard (1980) and Engle et al (1983).

We begin by setting out the basic framework and thus consider an n-dimensional VAR-ECM (p) process $\{X_t\}$, generated by

$$\Delta X_t = \sum_{i=1}^{P-1} \Gamma_i \Delta X_{t-i} + \alpha \beta' X_{t-1} + \varepsilon_t, \ t = 1,..,T \qquad (1)$$

where Γ_i α, β are, respectively (n, n), (n, r), (n, r), $0 < r < n$ matrices such that $\Pi = \alpha \beta'$; ε_t is an i.i.d normal distributed vector of errors, with a zero mean and a positive definite covariance matrix Σ; and p is a constant integer. To keep the notation as simple as possible, we omit deterministic components.

It is assumed in addition that (i) $\left|(I_n - \sum_{i=1}^{P-1} \Gamma_i z^i)(1 - z) + \alpha \beta' z\right| = 0$ implies either $|z| > 1$ or $z = 1$, and that (ii) the matrix $\alpha'_\perp (I_n - \sum_{i=1}^{p} \Gamma_i) \beta_\perp$ is invertible, where $\alpha_\perp$ and $\beta_\perp$ are both full rank (n, n-r) matrices satisfying $\alpha' \alpha_\perp = 0$ and $\beta' \beta_\perp = 0$. These two conditions ensure that $\{X_t\}$, and $\{\beta' X_t\}$, are respectively I (1) and I (0) and that the Granger theorem (1987) is satisfied.

Let us now consider the partition of X_t, Γ_i, α and β respectively into $\begin{pmatrix} Y_t \\ Z_t \end{pmatrix}$, $\begin{bmatrix} \Gamma_{YY,i} & \Gamma_{YZ,i} \\ \Gamma_{ZY,i} & \Gamma_{ZZ,i} \end{bmatrix}$, $\begin{bmatrix} \alpha_Y \\ \alpha_Z \end{bmatrix}$, $\begin{bmatrix} \beta_Y \\ \hline \beta_Z \end{bmatrix}$, $Y_t \in R^g$, $Z_t \in R^k$, with $g + k = n$, where Y_t is a vector of g variables that we shall call "endogenous", and Z_t a vector containing the k remaining other variables that we shall call "exogenous". Equation (1) can then easily be rewritten into one of the two following well-known forms :

Form 1 :

$$\begin{cases} \Delta Y_t = \sum_{i=1}^{P-1} \Gamma_{YY,i} \Delta Y_{t-i} + \sum_{i=1}^{P-1} \Gamma_{YZ,i} \Delta Z_{t-i} + \alpha_Y \begin{bmatrix} \beta'_Y & \| \beta'_Z \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ Z_{t-1} \end{bmatrix} + \varepsilon_{Y,t} & \mathbf{(2)} \\ \Delta Z_t = \sum_{i=1}^{P-1} \Gamma_{ZY,i} \Delta Y_{t-i} + \sum_{i=1}^{P-1} \Gamma_{ZZ,i} \Delta Z_{t-i} + \alpha_Z \begin{bmatrix} \beta'_Y & \| \beta'_Z \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ Z_{t-1} \end{bmatrix} + \varepsilon_{Z,t} & \mathbf{(3)} \end{cases}$$

with $\begin{pmatrix} \varepsilon_{Y,t} \\ \varepsilon_{Z,t} \end{pmatrix} \sim N\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix} \Sigma_{YY} & \Sigma_{YZ} \\ \Sigma_{ZY} & \Sigma_{ZZ} \end{pmatrix}\right].$

Form 2 :

$$\begin{cases} \Delta Y_t = \sum_{i=1}^{P-1} \Gamma_{YY,i}^{+} \Delta Y_{t-i} + \sum_{i=0}^{P-1} \Gamma_{YZ,i}^{+} \Delta Z_{t-i} + \alpha_Y^{+} \left[\beta_Y' \quad \| \beta_Z'\right] \begin{bmatrix} Y_{t-1} \\ Z_{t-1} \end{bmatrix} + \eta_{Y,t} & \textbf{(4)} \\ \textit{conditional model} & \\ \Delta Z_t = \sum_{i=1}^{P-1} \Gamma_{ZY,i} \Delta Y_{t-i} + \sum_{i=1}^{P-1} \Gamma_{ZZ,i} \Delta Z_{t-i} + \alpha_Z \left[\beta_Y' \quad \| \beta_Z'\right] \begin{bmatrix} Y_{t-1} \\ Z_{t-1} \end{bmatrix} + \varepsilon_{Z,t} & \textbf{(5)} \\ \textit{marginal model} & \end{cases}$$

with $\begin{cases} \Gamma_{YY}^{+}(L) = \Gamma_{YY}(L) - \Sigma_{YZ}\Sigma_{ZZ}^{-1}\Gamma_{ZY}(L) = Id_g - \sum_{i=1}^{P-1}\Gamma_{YY,i}^{+}L^i \\ \Gamma_{YZ}^{+}(L) = \Gamma_{YZ}(L) - \Sigma_{YZ}\Sigma_{ZZ}^{-1}\Gamma_{ZZ}(L) = -\sum_{i=0}^{P-1}\Gamma_{YZ,i}^{+}L^i \\ \alpha_Y^{+} = \alpha_Y - \Sigma_{YZ}\Sigma_{ZZ}^{-1}\alpha_Z \\ \eta_{Y,t} = \varepsilon_{Y,t} - \Sigma_{YZ}\Sigma_{ZZ}^{-1}\varepsilon_{Z,t} \\ \Sigma_{YY}^{+} = \Sigma_{YY} - \Sigma_{YZ}\Sigma_{ZZ}^{-1}\Sigma_{ZY} \end{cases}$ and $\begin{pmatrix} \eta_{Y,t} \\ \varepsilon_{Z,t} \end{pmatrix} \sim N\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix} \Sigma_{YY}^{+} & 0 \\ 0 & \Sigma_{ZZ} \end{pmatrix}\right].$

The first VAR-ECM form permits to explicit the parameters that appear respectively in the equations describing the evolution of the endogenous variables Y_t and those describing the evolution of the exogenous variables Z_t. The second form corresponds to the VAR-ECM block recursive form[5] and its main interest is to provide the analytic expression of the conditional error correction model. Note that equation (5) which corresponds to the marginal model is the same as equation (3) : this is because the residual orthogonalisation doesn't affect the equations describing the evolution of the Z_t variables.

III. Canonical decomposition of the Π matrix in the I(1) case and exogeneity conditions

[5] See Urbain (1992) or Rault (1997), for a complete derivation of the VAR-ECM block recursive form.

As it is well-known, the presence or lack of exogeneity depends upon what the parameters of interests are, but contrary to what is often assumed in a cointegrated framework, they need not necessarily include all the cointegrating vectors of the system. For instance when analysing VAR-ECM models or a conditional model, the parameters of interest might be only the cointegrating vectors that enter the conditional model itself for instance, or only those involving both endogenous and exogenous variables, or both short run and long run parameters of the conditional model,…To take these possibilities into account, we can now state the following result, that provides a canonical decomposition of the Π matrix in the I(1) case[6], which constitutes a suitable framework in which less restrictive exogeneity conditions can then be formulated [7].

THEOREM 1

Let $\Pi = \alpha\,\beta'$ be a (n, n) reduced rank matrix of rank r ($0 < r < n$) and partition β into $\begin{bmatrix}\beta_Y \\ \hline \beta_Z\end{bmatrix}$. If we define $r_1 = \text{rank}\,(\beta_Y)$ with $\max(0, r-k) \le r_1 \le \min(g, r)$ [8], then the α and β matrices can always be reparametrised as follows :

$$\beta = \lfloor \beta_1 \;\|\; \beta_2 \rfloor = \left[\begin{array}{c|c} \beta_{Y1} & 0 \\ \hline \beta_{Z1} & \beta_{Z2} \end{array}\right], \quad \alpha = \lfloor \alpha_1 \;\|\; \alpha_2 \rfloor = \left[\begin{array}{c|c} \alpha_{Y1} & \alpha_{Y2} \\ \hline \alpha_{Z1} & \alpha_{Z2} \end{array}\right],$$

$$\underset{r_1}{\longleftrightarrow}\;\; \underset{r-r_1}{\longleftrightarrow} \qquad\qquad \underset{r_1}{\longleftrightarrow}\;\; \underset{r-r_1}{\longleftrightarrow}$$

with $\underset{(n,r_1)}{\beta_1} = \begin{bmatrix} \underset{(g,r_1)}{\beta_{Y1}} \\ \hline \underset{(k,r_1)}{\beta_{Z1}} \end{bmatrix}$, $\underset{(n,r-r_1)}{\beta_2} = \begin{bmatrix} \underset{(g,r-r_1)}{0} \\ \hline \underset{(k,r-r_1)}{\beta_{Z2}} \end{bmatrix}$, $\underset{(n,r_1)}{\alpha_1} = \begin{bmatrix} \underset{(g,r_1)}{\alpha_{Y1}} \\ \hline \underset{(k,r_1)}{\alpha_{Z1}} \end{bmatrix}$, $\underset{(n,r-r_1)}{\alpha_2} = \begin{bmatrix} \underset{(g,r-r_1)}{\alpha_{Y2}} \\ \hline \underset{(k,r-r_1)}{\alpha_{Z2}} \end{bmatrix}$

Taking these new writings into account, the Π matrix can now be written as :

$$\Pi = \alpha\,\beta' = \left[\begin{array}{c|c} \alpha_{Y1}\beta'_{Y1} & \alpha_{Y1}\beta'_{Z1} + \alpha_{Y2}\beta'_{Z2} \\ \hline \alpha_{Z1}\beta'_{Y1} & \alpha_{Z1}\beta'_{Z1} + \alpha_{Z2}\beta'_{Z2} \end{array}\right].$$

The proof is given in appendix.

[6] This theorem has already been used in Rault (2000) for Granger non-causality testing from Y to Z in a cointegrated framework. Nevertheless to make the discussion clear and to enable a clear link with the empirical analysis performed in this paper, we restate this theorem.

[7] See Richard (1980), Engle and al (1983) for formal definitions of the weak and strong exogeneity hypotheses as well as for an extended discussion.

[8] This condition ensures that the β matrix is of rang r.

The theorem is proved using a basis change in the cointegrating space, which permits to separate the cointegrating vectors related to the exogenous variables alone, from those related to both endogenous and exogenous variables : r_1 is thus defined as the rank of β_{Y1} *and is invariant to any reparametrisation* $\alpha^* = \alpha\,(P')^{-1}$, $\beta^* = \beta\, P$, for any non-singular matrix P of dimension (r, r)[9]. To this partition corresponds a new α partitioned into $\lfloor \alpha_1 \;\|\; \alpha_2 \rfloor$. It must be noticed that our theorem implies no loss of generality, and only requires the determination of the r_1 rank of the β_Y matrix[10].

Given this canonical decomposition, equations 2 to 5 can be rewritten as :

Form 1' :

$$\begin{cases} \Delta Y_t = \sum_{i=1}^{P-1} \Gamma_{YY,i}\Delta Y_{t-i} + \sum_{i=1}^{P-1}\Gamma_{YZ,i}\Delta Z_{t-i} + \alpha_{Y1}\beta'_{Y1}Y_{t-1} + (\alpha_{Y1}\beta'_{Z1} + \alpha_{Y2}\beta'_{Z2})\,Z_{t-1} + \varepsilon_{Y,t} & \textbf{(2')} \\ \Delta Z_t = \sum_{i=1}^{P-1} \Gamma_{ZY,i}\Delta Y_{t-i} + \sum_{i=1}^{P-1}\Gamma_{ZZ,i}\Delta Z_{t-i} + \alpha_{Z1}\beta'_{Z1}Y_{t-1} + (\alpha_{Z1}\beta'_{Z1} + \alpha_{Z2}\beta'_{Z2})\,Z_{t-1} + \varepsilon_{Z,t} & \textbf{(3')} \end{cases}$$

Form 2' :

$$\begin{cases} \Delta Y_t = \sum_{i=1}^{P-1} \Gamma^+_{YY,i}\Delta Y_{t-i} + \sum_{i=0}^{P-1}\Gamma^+_{YZ,i}\Delta Z_{t-i} + \alpha^+_{Y1}\beta'_{Y1}Y_{t-1} + (\alpha^+_{Y1}\beta'_{Z1} + \alpha^+_{Y2}\beta'_{Z2})\,Z_{t-1} + \eta_{Y,t} & \textbf{(4')} \\ \textit{conditional model} & \\ \Delta Z_t = \sum_{i=1}^{P-1} \Gamma_{ZY,i}\Delta Y_{t-i} + \sum_{i=1}^{P-1}\Gamma_{ZZ,i}\Delta Z_{t-i} + \alpha_{Z1}\beta'_{Z1}Y_{t-1} + (\alpha_{Z1}\beta'_{Z1} + \alpha_{Z2}\beta'_{Z2})\,Z_{t-1} + \varepsilon_{Z,t} & \textbf{(5')} \\ \textit{marginal model} & \end{cases}$$

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[9] The proof is straightforward since $\beta^* = \begin{pmatrix}\beta_Y \\ \beta_Z\end{pmatrix} P = \begin{pmatrix}\beta_Y P \\ \beta_Z P\end{pmatrix} = \begin{pmatrix}\beta^*_Y \\ \beta^*_Z\end{pmatrix}$ implies rank (β^*_Y) = rank ($\beta_Y\, P$) = rank (β_Y) if the matrix P is invertible.

[10] The way this r_1 rank can be investigated is discussed in section 6.

with $\begin{cases} \Gamma^{+}_{YY}(L) = \Gamma_{YY}(L) - \Sigma_{YZ}\Sigma^{-1}_{ZZ}\Gamma_{ZY}(L) = Id_g - \sum_{i=1}^{P-1}\Gamma^{+}_{YY,i}L^i \\ \Gamma^{+}_{YZ}(L) = \Gamma_{YZ}(L) - \Sigma_{YZ}\Sigma^{-1}_{ZZ}\Gamma_{ZZ}(L) = -\sum_{i=0}^{P-1}\Gamma^{+}_{YZ,i}L^i \\ \alpha^{+}_{Y1} = \alpha_{Y1} - \Sigma_{YZ}\Sigma^{-1}_{ZZ}\alpha_{Z1} \\ \alpha^{+}_{Y2} = \alpha_{Y2} - \Sigma_{YZ}\Sigma^{-1}_{ZZ}\alpha_{Z2} \\ \eta_{Y,t} = \varepsilon_{Y,t} - \Sigma_{YZ}\Sigma^{-1}_{ZZ}\varepsilon_{Z,t} \\ \Sigma^{+}_{YY} = \Sigma_{YY} - \Sigma_{YZ}\Sigma^{-1}_{ZZ}\Sigma_{ZY} \end{cases}$ and $\begin{pmatrix}\eta_{Y,t} \\ \varepsilon_{Z,t}\end{pmatrix} \sim N\left[\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}\Sigma^{+}_{YY} & 0 \\ 0 & \Sigma_{ZZ}\end{pmatrix}\right].$

Then additional restrictions related to weak exogeneity hypothesis, non-causality hypothesis and strong exogeneity hypothesis for instance can easily be investigated. To this end, we may now state the following results :

Proposition 1. Sufficient weak exogeneity conditions for the parameters of the conditional model[11]

Let the variables be generated according to the VAR-ECM model (equations 2' and 3'), and define φ as the parameters of interest for the applied economist, then we say that Z_t is weakly exogenous for $\varphi = \text{vec}\,(\Gamma^{+}_{YY,1},\ldots,\Gamma^{+}_{YY,p-1};\Gamma^{+}_{YZ,1},\ldots,\Gamma^{+}_{YZ,p-1};\alpha^{+}_{Y1},\ \beta_1')$, namely both the long-run and short-run parameters of the conditional model (equation 4'), if $\begin{cases}\alpha_{Z1} = 0 \\ \alpha^{+}_{Y2} = 0\end{cases}$.

The proof is given in appendix.

It must be emphasised that these weak exogeneity conditions remain indeed the same if the parameters of interest are only given by the first r_1 cointegrating vectors containing both exogenous and endogenous variables.

If instead of being the parameters of equations 4', the parameters of the first g equations of the VAR-ECM model written in reduced form (equation 2') are of structural interest in an empirical application, then we have the following corollary :

[11] Let's remember that Engle and al 's (1983) define a vector of Z_t variables to be weakly-exogenous for the parameters of interest, if (i) the parameters of interest only depend on those of the conditional model, (ii) the

Corollary 1. Sufficient weak exogeneity conditions for the parameters of equation 2'

Z_t is weakly exogenous for φ = vec ($\Gamma_{YY,1},\ldots,\ \Gamma_{YY,p-1};\ \Gamma_{YZ,1},\ldots,\ \Gamma_{YZ,p-1};\ \alpha_{Y1},\ \beta_1'$), namely both the long-run and short-run parameters of equation 2', if : $\begin{cases} \alpha_{Z1} = 0 \\ \alpha_{Y2} = 0 \\ \Sigma_{YZ} = 0 \end{cases}$.

The proof is given in appendix.

It must be noticed that our weak exogeneity conditions don't prohibit the marginal model from including cointegration relations that involve the Z_t alone and thus don't entail this model to be a VAR in first difference as the usual weak exogeneity conditions do [12]. Moreover as a subset of cointegrating vectors is considered to be of primarily interest for investigators, namely the cointegrating vectors containing both exogenous and endogenous variables, only the corresponding part of α is required to vanish for weak exogeneity. Thus the conditional and marginal model contains separate sets of cointegrating vectors, and Z_t is weakly exogenous for the cointegrating vectors entering the conditional model, which means in other words that we can make efficient inference on the parameters of interest in the conditional model or partial system, and neglect the marginal model without a loss of information. It should however be stressed that incorrect exogeneity assertions invalidate any subsequent inference so that testing for exogeneity assumption should be an integral and unavoidable step in any partial system modelling exercise.

Proposition 2. Necessary and sufficient non-causality conditions for the parameters of equation 2'

Let the variables be generated according to the VAR-ECM model (equations 2' and 3'), then we say that Y doesn't cause Z in Granger sense (1969), if and only if : $\begin{cases} \Gamma_{ZY,i} = 0, i = 1,\ldots,p-1 \\ \alpha_{Z1} = 0 \end{cases}$.

The proof is given in appendix.

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parameters of the conditional and marginal models (φ^{+}_{Y} and φ_Z) are variation free, i.e there exists a sequential cut of the two parameters spaces for φ^{+}_{Y} and φ_Z (Florens and Mouchart, 1980).

[12] If the parameters of interests are for instance all the long run parameters of the traditional conditional model, namely the cointegrating vectors and the error correction coefficients of equation 4, then a sufficient condition for weak exogeneity of Z_t is $\alpha_Z = 0$, Johansen (1992a, 1992b), Urbain (1992) : this implies that the conditioning variables are not error-correcting.

Note that these conditions remain the same if one wishes to investigate non-causality from Y to Z in the orthogonalised VAR-ECM (equation 4' and 5'). Furthermore, it must be underlined that non-causality test statistics in VAR-ECM models usually require special conditions to be asymptotically χ^2 distributed : this is because long run non causality from Y to Z $\{\alpha_Z \beta'_Y = 0\}$ implies non linear constraints on long run parameters. What differs here from the usual case, is that given the canonical decomposition proposed in theorem 1, long run non causality hypothesis reduces to $\alpha_{Z1} = 0$ (since β_{Y1} is of full rank column r_1), and hence can be tested using asymptotic χ^2 tests[13] .

Proposition 3. Sufficient strong exogeneity conditions for the parameters of the conditional model

Let the variables be generated according to the VAR-ECM model (equations 2' and 3'), and define φ as the parameters of interest for the applied economist, then we say that Z_t is strongly exogenous for $\varphi = \text{vec}\,(\Gamma^+_{YY,1},\ldots,\ \Gamma^+_{YY,p-1};\ \Gamma^+_{YZ,1},\ldots,\ \Gamma^+_{YZ,p-1};\ \alpha^+_{Y1},\ \beta_1')$, namely both the long-run and short-run parameters of the conditional model (equation 4'), if

$$\begin{cases} \alpha_{Z1} = 0 \\ \alpha^+_{Y2} = 0 \\ \Gamma_{ZY,i} = 0,\ i = 1,\ldots p-1 \end{cases}.$$

The proof is given in appendix.

In the case when the parameters of structural interest are those of the first g equations of the VAR-ECM model written in reduced form (equation 2') we have as previously the following corollary :

[13] See Rault (1999) for further details.

Corollary 2. Sufficient strong exogeneity conditions for the parameters of equation 2'

Z_t is strongly exogenous for φ = vec ($\Gamma_{YY,1},\ldots, \Gamma_{YY,p-1}; \Gamma_{YZ,1},\ldots, \Gamma_{YZ,p-1}; \alpha_{Y1}, \beta_1'$), namely both the long-run and short-run parameters of equation 2', if :

$$\begin{cases} \alpha_{Z1} = 0 \\ \alpha_{Y2} = 0 \\ \Sigma_{YZ} = 0 \\ \Gamma_{ZY,i} = 0, i = 1,\ldots p-1 \end{cases}.$$

The proof is given in appendix.

Strong exogenous hypothesis for the parameters of interest combine weak exogeneity with non-causality and entails a complete separation of the process generating $\{Y_t / Z_t\}$ and $\{Z_t\}$, that is between the conditional model and the marginal one. Its main interest is to permit valid forecasts of Y from the conditional model, given forecasts of Z from the marginal model.

IV. A more general canonical decomposition of the Π matrix and exogeneity conditions

The exogeneity conditions formulated in the previous section have been shown to be less restrictive than the usual exogeneity ones. However it can still truly be argued that we have implicitly assumed that only the long-run relations involving both endogenous and exogenous variables were of potential interest for the investigator, i.e that the conditional model only contains such relations. Actually one could easily face the possibility that VAR-ECM models or conditional models also contain long-run relations involving the Z_t variables alone, which may also have a potential interest for investigators in specific empirical applications : in that case exogeneity requires that such purely exogenous long run paths don't appear in the marginal model. To that end, we propose in this section a more general decomposition of the long run matrix Π, which can then constitute a suitable framework in which we formulate more general exogeneity conditions, allowing both the marginal and conditional model to contain purely exogenous long-run paths. These conditions can give rise to I(2) behaviour of the endogenous variables Y in case of non-causality from Y to Z.

Before stating the new decomposition of the Π matrix, let us first relax assumption (ii) of section II, i.e we do not assume that the matrix $\alpha'_{\perp} (I_n - \sum_{i=1}^{p} \Gamma_i) \beta_{\perp}$ is non singular any more, where $\alpha_{\perp}$ and $\beta_{\perp}$ are both full rank (n, n-r) matrices satisfying $\alpha' \alpha_{\perp} = 0$ and $\beta' \beta_{\perp} = 0$: this means in other words that we do not preclude the possibility that some X_t elements are integrated of order two [14]. Then we have the following result :

[14] Johansen (1995) has shown that if this matrix has reduced rank, then some elements of X_t are I (2).

THEOREM 2

Let $\Pi = \alpha \beta'$ be a (n, n) reduced rank matrix of rank r (0 < r < n) and consider the reparametrisation $\beta = \left[\frac{\beta_{Y1}}{\beta_{Z1}} \Bigg\| \frac{0}{\beta_{Z2}}\right]$ and $\alpha = \left[\frac{\alpha_{Y1}}{\alpha_{Z1}} \Bigg\| \frac{\alpha_{Y2}}{\alpha_{Z2}}\right]$ given by theorem 1, then :

(i) there exists an integer r_2 so that the α and β matrices can always be reparametrised as follows :

$$\alpha = \left[\alpha_1 \;\|\; \alpha_{21} \;\|\; \alpha_{22}\right] = \Bigg[\underbrace{\frac{\alpha_{Y1}}{\alpha_{Z1}}}_{\longleftrightarrow\; r_1} \Bigg\| \underbrace{\frac{\alpha_{Y21}}{0}}_{\longleftrightarrow\; r^*} \Bigg\| \underbrace{\frac{\alpha_{Y22}}{\alpha_{Z22}}}_{\longleftrightarrow\; r_2}\Bigg],$$

$$\beta = \left[\beta_1 \;\|\; \beta_{21} \;\|\; \beta_{22}\right] = \Bigg[\underbrace{\frac{\beta_{Y1}}{\beta_{Z1}}}_{\longleftrightarrow\; r_1} \Bigg\| \underbrace{\frac{0}{\beta_{Z21}}}_{\longleftrightarrow\; r^*} \Bigg\| \underbrace{\frac{0}{\beta_{Z22}}}_{\longleftrightarrow\; r_2}\Bigg],$$

with $r_1 + r_2 + r^* = r$, α_{Z22} of rank $r_2 \geq 0$ ($\max(0, r-k) \leq r_1 + r_2 \leq \min(g, r)$)[15] and where,

$$\underset{(n,r_1)}{\alpha_1} = \begin{bmatrix} \overset{(g,r_1)}{\alpha_{Y1}} \\ \hline \underset{(k,r_1)}{\alpha_{Z1}} \end{bmatrix}, \quad \underset{(n,r^*)}{\alpha_{21}} = \begin{bmatrix} \overset{(g,r^*)}{\alpha_{Y21}} \\ \hline \underset{(k,r^*)}{0} \end{bmatrix}, \quad \underset{(n,r_2)}{\alpha_{22}} = \begin{bmatrix} \overset{(g,r_2)}{\alpha_{Y22}} \\ \hline \underset{(k,r_2)}{\alpha_{Z22}} \end{bmatrix},$$

$$\underset{(n,r_1)}{\beta_1} = \begin{bmatrix} \overset{(g,r_1)}{\beta_{Y1}} \\ \hline \underset{(k,r_1)}{\beta_{Z1}} \end{bmatrix}, \quad \underset{(n,r^*)}{\beta_{21}} = \begin{bmatrix} \overset{(g,r^*)}{0} \\ \hline \underset{(k,r^*)}{\beta_{Z21}} \end{bmatrix}, \quad \underset{(n,r_2)}{\beta_{22}} = \begin{bmatrix} \overset{(g,r_2)}{0} \\ \hline \underset{(k,r_2)}{\beta_{Z22}} \end{bmatrix}.$$

Taking these new writings into account, the Π matrix can now be written as :

$$\Pi = \left[\frac{\alpha_{Y1}\beta'_{Y1}}{\alpha_{Z1}\beta'_{Y1}} \Bigg\| \frac{\alpha_{Y1}\beta'_{Z1} + \alpha_{Y21}\beta'_{Z21} + \alpha_{Y22}\beta'_{Z22}}{\alpha_{Z1}\beta'_{Z1} + \alpha_{Z22}\beta'_{Z22}}\right].$$

(ii) if in addition $\alpha_{Z1} = 0$, then r_2 is uniquely defined and is invariant to the chosen reparametrisation.

The proof is given in appendix.

[15] This condition ensures that the α matrix is of rang r.

The theorem is proved using a second basis change in the dual space of the cointegrating space, namely in the adjustment space generated by the α rows, which leads to partitioning the $r - r_1$ long run relations involving the Z_t variables alone into two sub-groups of respectively dimension r^* and r_2 : a first sub-group which only belongs to the conditional model and a second one which both belong to the marginal and conditional models (if we choose to apply this canonical decomposition to the VAR-ECM block-recursive form (equations 4 and 5)). To this second partition on the α adjustment space corresponds a new β whose representation is given by theorem 2. The fact that the α and β matrices are of rank r entails that the α_{Y21} and β_{Z21} matrices are of rank $r^* \geq 0$ and that β_{Z22} is of rank $r_2 \geq 0$. It is in particular necessary that $\begin{cases} r^* \leq \min(g,k) \\ r_2 \leq k \end{cases}$.

Given this canonical decomposition, equations 2 to 5 can be rewritten as :

Form 1'' :

$$\begin{cases} \Delta Y_t = \sum_{i=1}^{P-1} \Gamma_{YY,i} \Delta Y_{t-i} + \sum_{i=1}^{P-1} \Gamma_{YZ,i} \Delta Z_{t-i} + \alpha_{Y1} \beta'_{Y1} Y_{t-1} + (\alpha_{Y1}{}' \beta'_{Z1} + \alpha_{Y21} \beta'_{Z21} + \alpha_{Y22} \beta'_{Z22}) Z_{t-1} + \varepsilon_{Y,t} & \textbf{(2'')} \\ \Delta Z_t = \sum_{i=1}^{P-1} \Gamma_{ZY,i} \Delta Y_{t-i} + \sum_{i=1}^{P-1} \Gamma_{ZZ,i} \Delta Z_{t-i} + \alpha_{Z1} \beta'_{Y1} Y_{t-1} + (\alpha_{Z1} \beta'_{Z1} + \alpha_{Z22} \beta'_{Z22}) Z_{t-1} + \varepsilon_{Z,t} & \textbf{(3'')} \end{cases}$$

Form 2'' :

$$\begin{cases} \Delta Y_t = \sum_{i=1}^{P-1} \Gamma^+_{YY,i} \Delta Y_{t-i} + \sum_{i=0}^{P-1} \Gamma^+_{YZ,i} \Delta Z_{t-i} + \alpha^+_{Y1} \beta'_{Y1} Y_{t-1} + (\alpha^+_{Y1}{}' \beta'_{Z1} + \alpha^+_{Y21} \beta'_{Z21} + \alpha^+_{Y22} \beta'_{Z22}) Z_{t-1} + \eta_{Y,t} & \textbf{(4'')} \\ \textit{conditional model} & \\ \Delta Z_t = \sum_{i=1}^{P-1} \Gamma_{ZY,i} \Delta Y_{t-i} + \sum_{i=1}^{P-1} \Gamma_{ZZ,i} \Delta Z_{t-i} + \alpha_{Z1} \beta'_{Y1} Y_{t-1} + (\alpha_{Z1} \beta'_{Z1} + \alpha_{Z22} \beta'_{Z22}) Z_{t-1} + \varepsilon_{Z,t} & \textbf{(5'')} \\ \textit{marginal model} & \end{cases}$$

with $\begin{cases} \Gamma^{+}_{YY}(L) = \Gamma_{YY}(L) - \Sigma_{YZ}\Sigma^{-1}_{ZZ}\Gamma_{ZY}(L) = Id_g - \sum_{i=1}^{P-1}\Gamma^{+}_{YY,i}L^i \\ \Gamma^{+}_{YZ}(L) = \Gamma_{YZ}(L) - \Sigma_{YZ}\Sigma^{-1}_{ZZ}\Gamma_{ZZ}(L) = -\sum_{i=0}^{P-1}\Gamma^{+}_{YZ,i}L^i \\ \alpha^{+}_{Y1} = \alpha_{Y1} - \Sigma_{YZ}\Sigma^{-1}_{ZZ}\alpha_{Z1} \\ \alpha^{+}_{Y21} = \alpha_{Y21} \\ \alpha^{+}_{Y22} = \alpha_{Y22} - \Sigma_{YZ}\Sigma^{-1}_{ZZ}\alpha_{Z22} \\ \eta_{Y,t} = \varepsilon_{Y,t} - \Sigma_{YZ}\Sigma^{-1}_{ZZ}\varepsilon_{Z,t} \\ \Sigma^{+}_{YY} = \Sigma_{YY} - \Sigma_{YZ}\Sigma^{-1}_{ZZ}\Sigma_{ZY} \end{cases}$ and $\begin{pmatrix}\eta_{Y,t} \\ \varepsilon_{Z,t}\end{pmatrix} \sim N\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\Sigma^{+}_{YY} & 0 \\ 0 & \Sigma_{ZZ}\end{pmatrix}\right]$.

The table below summarises in which equations these three distinctive groups of long-run relations appear, in the case when this canonical decomposition is applied to the VAR-ECM block-recursive form (equations 4 and 5):

Cointegrating relations ($r_1 + r_2 + r^* = r$)	Coefficients in the conditional model	Coefficients in the marginal model
$[\beta'_{Y1}, \beta'_{Z1}]\begin{bmatrix}Y_t \\ Z_t\end{bmatrix} = \eta_{1t} \sim I(0)$ in number r_1	α^{+}_{Y1}	α_{Z1}
$\beta'_{Z21} Z_t = \eta^{*}_{2t} \sim I(0)$ in number r^*	α^{+}_{Y21}	0
$\beta'_{Z22} Z_t = \eta_{2t} \sim I(0)$ in number r_2	α^{+}_{Y22}	α_{Z22}

N.B :

• The canonical decomposition given in theorem 2 can be applied to any singular Π matrix of rank r.

• Some rank conditions are explicitly given in theorem 2. Nevertheless one must keep in mind that the ranks of the different matrix blocks are yet always linked by the two following conditions :

- the $\Pi = \alpha\beta'$ matrix is of rank r,
- the $\alpha'_{\perp}(I_n - \sum_{i=1}^{p}\Gamma_i)\beta_{\perp}$ is of full rank in the I(1) case and of reduced rank in the I(2) case.

Consequently the different parameter blocks cannot be equal to zero independently of the short-run coefficients Γ, and of the rank conditions given below.

Then additional restrictions related to weak exogeneity hypothesis, non-causality hypothesis and strong exogeneity hypothesis for instance can as in section III easily be investigated[16]. To this end, we may now state the following results :

Proposition 4. Necessary and sufficient weak exogeneity conditions for the parameters of the conditional model

Let the variables be generated according to the VAR-ECM model (equations 2'' and 3''), and define φ as the parameters of interest for the applied economist, then we say that Z_t is weakly exogenous for φ = vec ($\Gamma^{+}_{YY,1},\ldots, \Gamma^{+}_{YY,p-1}$; $\Gamma^{+}_{YZ,1},\ldots, \Gamma^{+}_{YZ,p-1}$; α^{+}_{Y1}, β_1', α^{+}_{Y21}, β'_{Z21}), namely both the long-run and short-run parameters of the conditional model (equation 4''), if and only if : $\begin{cases} \alpha_{Z1} = 0 \\ \alpha^{+}_{Y22} = 0 \end{cases}$.

The proof is given in appendix.

It must be emphasised that these weak exogeneity conditions remain indeed the same if the parameters of interest are only given by respectively the r_1 cointegrating vectors containing both exogenous and endogenous variables and by the r*. cointegration relations that involve the Z_t alone.

If instead of being the parameters of equations 4'', the parameters of the first g equations of the VAR-ECM model written in reduced form (equation 2'') are of structural interest in an empirical application, then we have the following corollary :

[16] It must be noticed that these exogeneity conditions are all presented in the canonical decomposition of the Π matrix given in theorem 2 (i) and not (ii) in order to entail no loss of generality and to show that $\alpha_{Z1} = 0$ is a necessary and sufficient condition.

Corollary 3. Necessary and sufficient weak exogeneity conditions for the parameters of equation 2'

Z_t is weakly exogenous for φ = vec ($\Gamma_{YY,1},\ldots, \Gamma_{YY,p-1}; \Gamma_{YZ,1},\ldots, \Gamma_{YZ,p-1}; \alpha_{Y1}, \beta_1'$, $\alpha_{Y21}, \beta'_{Z21}$), namely both the long-run and short-run parameters of equation 2'', if and only if :

$$\begin{cases} \alpha_{Z1} = 0 \\ \alpha_{Y22} = 0 \\ \Sigma_{YZ} = 0 \end{cases} .$$

The proof is given in appendix.

As in section III, our weak exogeneity conditions don't prohibit the marginal model from including cointegration relations that involve the Z_t alone and thus don't entail this model to be a VAR in first difference as the usual weak exogeneity conditions do. What is more, these conditions can be seen to be more general than the previous ones, since the conditional and marginal models now contain two completely separate sets of purely exogenous long run paths. This means that the investigator is now offered the possibility of also being interested in one subset of purely exogenous long run paths in addition to the long run relations involving both endogenous and exogenous variables.

Thus weak exogeneity for instance for all parameters of the conditional model (equation 4''), implies that :

(i) none of the r_1 long-run relations containing both endogenous and exogenous variables appear in the marginal model,

(ii) none of the r_2 purely exogenous long-run relations appearing in the marginal model, belong to the conditional model.

Note that as the second basis change entails no modifications of the first r_1 long run relations and their weights, Granger non-causality conditions from Y to Z remain the same as in section III and thus can be tested using asymptotic χ^2 tests.

N.B : If X_t is a process integrated of order 1, then the necessary and sufficient non-causality conditions entail $r^* = 0$. In fact if r^* were strictly positive, the non-causality conditions given above would be sufficient for the existence of variables integrated of order 2, since the matrix

$\alpha'_{\perp}$ (I_n - $\sum_{i=1}^{p}$ Γ_i) $\beta_{\perp}$ would have in this case reduced rank (see Mosconi et Giannini [1992], theorem 2).

Proposition 5. Necessary and sufficient strong exogeneity conditions for the parameters of the conditional model

Let the variables be generated according to the VAR-ECM model (equations 2'' and 3''), and define φ as the parameters of interest for the applied economist, then we say that Z_t is strongly exogenous for φ = vec ($\Gamma^{+}_{YY,1}$,…, $\Gamma^{+}_{YY,p-1}$; $\Gamma^{+}_{YZ,1}$,…, $\Gamma^{+}_{YZ,p-1}$; α^{+}_{Y1}, β_1', α^{+}_{Y21}, β'_{Z21}), namely both the long-run and short-run parameters of the conditional model (equation 4''), if and only if :

$$\begin{cases} \alpha_{Z1} = 0 \\ \alpha^{+}_{Y22} = 0 \\ \Gamma_{ZY,i} = 0,\ i = 1,\dots p-1 \end{cases}$$

The proof is given in appendix.

In the case when the parameters of structural interest are those of the first g equations of the VAR-ECM model written in reduced form (equation 2') we have as previously the following corollary :

Corollary 4. Necessary and sufficient strong exogeneity conditions for the parameters of equation 2'

Z_t is strongly exogenous for φ = vec ($\Gamma_{YY,1}$,…, $\Gamma_{YY,p-1}$; $\Gamma_{YZ,1}$,…, $\Gamma_{YZ,p-1}$; α_{Y1}, β_1', α_{Y21}, β'_{Z21}), namely both the long-run and short-run parameters of equation 2'', if and only if :

$$\begin{cases} \alpha_{Z1} = 0 \\ \alpha_{Y22} = 0 \\ \Sigma_{YZ} = 0 \\ \Gamma_{ZY,i} = 0,\ i = 1,\dots p-1 \end{cases} .$$

The proof is given in appendix.

V. Inference and testing

As pointed out in the previous sections, theorems 1 and 2 require the determination of two specific ranks of sub-matrices, denoted r_1 and r_2. These ranks may be investigated using the two following stages :

1) first, determine the r_1 rank of the β_Y matrix which corresponds to the reparametrisation of α and β given in theorem 1, using the simple sequential test procedure proposed by Rault (2000), which is based on asymptotically χ^2 distributed LR tests and whose properties have been analysed with Monte-Carlo experiments.

2) then for the value of the r_1 rank found in stage 1, proceed as described in the following two steps :

(a) test $H_0 : \alpha_{Z1} = 0$ using conventional asymptotic χ^2 tests (see for instance Toda and Phillips (1991), (1993)).

If you accept the null hypothesis pass to step (b) or else stop. The latter case implies you also refuse weak and strong exogeneity hypotheses, as well as non-causality hypothesis from Y to Z, since as stated in the previous propositions, these hypotheses all require $\alpha_{Z1} = 0$.

(b) Consider the model defined by the restrictions

$$\beta = (\Psi_1, H_1\, \varphi_1, H_1\, \varphi_2), \text{ and } \alpha = (H_2\, \varphi_3, H_2\, \varphi_4, \Psi_2),$$

where H_1 and H_2 are respectively (N, s_1) and (N, s_2) known matrices and φ_1, φ_2, φ_3, Ψ_1 , Ψ_2, are respectively (s_1, r_a), (s_1, r_c), (s_2, r_a), (N, r_a), (N, r_b) matrices of parameters to be estimated, with $r_b \leq s_1 \leq N$, $r_c \leq s_1 \leq N$, $r_a \leq s_2 \leq N$, $r_b \leq s_2 \leq N$, $r_a + r_b + r_c = r$. This test which only implies linear restrictions on the long-run parameters turns out to be asymptotically χ^2 distributed and for detailed discussions the reader is referred to Johansen and Juselius (1992) and (1995). As our aim is to determine the r_2 rank of the α_{Z2} sub-matrix, given that the

β_Y matrix is of rank r$_1$ and that $\alpha_{Z1} = 0$ (which ensure that r$_2$ is an invariant), we consider H$_1$ and H$_2$ matrices of the form $\mathrm{H}_1 = \begin{pmatrix} \underset{(g,k)}{0} \\ I_k \end{pmatrix}$ and $\mathrm{H}_1 = \begin{pmatrix} I_g \\ \underset{(k,g)}{0} \end{pmatrix}$.

More precisely, let us define m$_1$ = min (g, r), m$_2$ = max (0, r-k) and consider the following sequences of null hypotheses[17] :

$$\begin{cases} \text{- } \mathrm{H}_{0,1} : \{ \text{there exists a basis of the cointegrating and adjustment space so that } \beta = (\psi_1, \mathrm{H}_1\varphi_1, \mathrm{H}_1\varphi_2) \text{ and } \alpha = (\mathrm{H}_2\varphi_3, \mathrm{H}_2\varphi_4, \psi_2) \\ \text{with } \mathrm{r_a} = r_1, \mathrm{r_b} = 1, \mathrm{r_c} = r - \mathrm{r_a} - \mathrm{r_b}, \textit{namely} \begin{cases} \text{rank}(\beta_\mathrm{Y}) = \mathrm{r}_1 \\ \alpha_{Z1} = 0 \\ \text{rank}(\alpha_{Z2}) \le \mathrm{m}_1 \text{ -1} \end{cases} \text{with } \mathrm{m}_2 \le \text{rank}(\alpha_{Z2}) \le m_1 \}. \\ \text{- } \mathrm{H}_{0,2} : \{ \text{there exists a basis of the cointegrating and adjustment space so that } \beta = (\psi_1, \mathrm{H}_1\varphi_1, \mathrm{H}_1\varphi_2) \text{ and } \alpha = (\mathrm{H}_2\varphi_3, \mathrm{H}_2\varphi_4, \psi_2) \\ \text{with } \mathrm{r_a} = r_1, \mathrm{r_b} = 2, \mathrm{r_c} = r - \mathrm{r_a} - \mathrm{r_b}, \textit{namely} \begin{cases} \text{rank}(\beta_\mathrm{Y}) = \mathrm{r}_1 \\ \alpha_{Z1} = 0 \\ \text{rank}(\alpha_{Z2}) \le \mathrm{m}_1 \text{ - 2} \end{cases} \text{with } \mathrm{m}_2 \le \text{rank}(\alpha_{Z2}) \le m_1 \}. \\ \text{- } \mathrm{H}_{0,j} : \{ \text{there exists a basis of the cointegrating and adjustment space so that } \beta = (\psi_1, \mathrm{H}_1\varphi_1, \mathrm{H}_1\varphi_2) \text{ and } \alpha = (\mathrm{H}_2\varphi_3, \mathrm{H}_2\varphi_4, \psi_2) \\ \text{with } \mathrm{r_a} = r_1, \mathrm{r_b} = j, \mathrm{r_c} = r - \mathrm{r_a} - \mathrm{r_b}, \textit{namely} \begin{cases} \text{rank}(\beta_\mathrm{Y}) = \mathrm{r}_1 \\ \alpha_{Z1} = 0 \\ \text{rank}(\alpha_{Z2}) \le \mathrm{m}_1 \text{ - j} \end{cases} \text{with } \mathrm{m}_2 \le \text{rank}(\alpha_{Z2}) \le m_1 \}. \end{cases}$$

To test these different hypotheses, we adopt the following sequential test procedure [18]:

$$\begin{cases} \text{- Step 1: test } \mathrm{H}_{0,1} \text{ with the statistic } \psi_1 \text{ at the } \alpha_1 \textit{ level} \text{ and refuse } \mathrm{H}_{0,1} \text{ if } \psi_1 \succ \chi^2_{1-\alpha_1}(\mathrm{v}_1) \Rightarrow \begin{cases} \text{rank}(\beta_\mathrm{Y}) = \mathrm{r}_1 \\ \alpha_{Z1} = 0 \\ \text{rank}(\alpha_{Z2}) = \mathrm{m}_1 \end{cases} . \\ \text{- Step 2: test } \mathrm{H}_{0,2} \text{ with the statistic } \psi_2 \text{ at the } \alpha_2 \textit{ level} \text{ and refuse } \mathrm{H}_{0,2} \text{ if } \psi_2 \succ \chi^2_{1-\alpha_2}(\mathrm{v}_2) \Rightarrow \begin{cases} \text{rank}(\beta_\mathrm{Y}) = \mathrm{r}_1 \\ \alpha_{Z1} = 0 \\ \text{rank}(\alpha_{Z2}) = \mathrm{m}_1 \text{ -1} \end{cases} . \\ \text{- Step j: test } \mathrm{H}_{0,j} \text{ with the statistic } \psi_\mathrm{j} \text{ at the } \alpha_\mathrm{j} \textit{ level} \text{ and refuse } \mathrm{H}_{0,j} \text{ if } \psi_\mathrm{j} \succ \chi^2_{1-\alpha_\mathrm{j}}(\mathrm{v_j}) \Rightarrow \begin{cases} \text{rank}(\beta_\mathrm{Y}) = \mathrm{r}_1 \\ \alpha_{Z1} = 0 \\ \text{rank}(\alpha_{Z2}) = 1 \end{cases} ., \\ \qquad \text{else if } \psi_\mathrm{j} \prec \chi^2_{1-\alpha_\mathrm{j}}(\mathrm{v_j}) \Rightarrow \begin{cases} \text{rank}(\beta_\mathrm{Y}) = \mathrm{r}_1 \\ \alpha_{Z1} = 0 \\ \text{rank}(\alpha_{Z2}) = 0 \end{cases} . \\ \textit{for } j = 1, \ldots, m_1 - 1 \end{cases}$$

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[17] This procedure is a natural extension of the one presented in great details in Rault (1999).

Note that we only pass to step j if the null hypothesis $H_{0,\,j-1}$ has been accepted in step j-1. Each statistic is a likelihood ratio test :

$$\Psi_j = -2 \ln Q\,(H_j / H_1) = T\,[\sum_{i=1}^{ra} \ln(1 - \hat{\rho}_{1j}) + \sum_{i=1}^{rb} \ln(1 - \hat{\rho}_{2j}) + \sum_{i=1}^{rc} \ln(1 - \hat{\rho}_{3j}) - \sum_{i=1}^{r} \ln(1 - \tilde{\lambda}_j)]$$

which is asymptotically distributed as $\chi^2_{1-\alpha_j}(v_j)$ with $v_j = r_a (N - s_2 - 1) + r_b (2N - s_1 - s_2) + r_c (N - s_1 - 1)$ degrees of freedom : H_1 corresponds to the cointegrating hypothesis $\Pi = \alpha\,\beta'$, and $\tilde{\lambda}_j$, $\hat{\rho}_{1j}$, $\hat{\rho}_{2j}$, $\hat{\rho}_{3j}$ denotes the eigenvalues of respectively the unrestricted VAR-ECM, and the r_a, r_b, r_c restricted long run relations and their weights.

Once these r_1 and r_2 ranks have been determined, additional restrictions related to weak exogeneity hypothesis, non-causality hypothesis and strong exogeneity hypothesis as those developed in sections IV and V can be investigated. It must be emphasised that these exogeneity conditions express themselves as minimal conditions on the parameters on a canonical decomposition of the Π matrix in which the nullity of some parameters blocks don't imply any loss of generality and only entail linear restrictions on some parameters matrices. That's why these hypotheses can be tested using conventional asymptotic χ^2 distributed test statistics proposed by Johansen and Juselius (1992), Johansen (1995), Urbain (1992), or Boswijk (1992) for instance. Moreover the different test statistics can easily be computed in empirical applications using the PC-FIML software (1994, 1997), or CATS in RATS software for instance which make this a simple strategy to implement, since they allow to test general restrictions both on short-run and on long-run parameters of a VAR-ECM model : this will be shown in the empirical application of section VI.

VI. Exchange rate estimation for developing countries : the case of Mexico

The example chosen to illustrate the exogeneity conditions developed in this paper is an extension of the analysis contained in Avallone and Rault (1999) and concerns equilibrium real exchange rate evaluation for developing countries, based on the Edwards model (1994)

[18] Let's point out that this procedure performs quite well in term of α_{Z2} rank selection, even if some α_{Z2} columns are linear combination of others.

and on Elbadawi works (1994, 1997). The main difference with Avallone and Rault (1999) is that we shall here focus our analysis on only one emergent country (Mexico).

5.1 The economic background and the data used

Following Edwards' theoretical model (1994) which is the most extensively used framework in the object of equilibrium real exchange rate evaluation for developing countries, there should exist a unique long-run equilibrium relation between real exchange rate and its fundamental of the form :

$$\hat{p}_N = f\left(D_N , G_N , T, D_M , Y_X , v, Y \right)$$

Table 1 presents these theoretical determinants and reports the proxies that will be used afterwards in the econometric estimations.

TABLE 1 - Theoretical determinants and retained proxies

<table>
<tr><th></th><th>Theoretical determinants</th><th>Proxies in the econometric model</th></tr>
<tr><td>D_N</td><td>Private expenditures on non traded goods</td><td>A : private absorption</td></tr>
<tr><td>G_N</td><td>Public expenditures on non traded goods</td><td>G : public expenditures</td></tr>
<tr><td>T</td><td>Customs duties</td><td rowspan="4">OPEN : openness degree of the economy
TOT : terms of trade
EUV : exports unit value</td></tr>
<tr><td>D_M</td><td>Imports</td></tr>
<tr><td>Y_X</td><td>Exportable goods production</td></tr>
<tr><td>v</td><td>Export subsidies</td></tr>
<tr><td>Y</td><td>Domestic income</td><td>GDP : real GDP per capita</td></tr>
</table>

The theoretical determinants of equilibrium exchange rate (hereafter ERER) are measured in the subsequent way. The four T, D_M, Y_X and v variables concern external equilibrium. They describe the structural ability of the economy to insure its trade balance thus we choose to measure them by the three following variables: *OPEN*, *TOT* and *EUV*. The *OPEN* variable is defined as the sum of exports and imports divided by *GDP*. This indicator gives information on the trade policy conducted by the economy. Commercial liberalisation translates into

import tariffs and export subsidy reduction. It tends to provoke a trade balance deterioration that requires ERER depreciation. Concerning the influence of *TOT* on ERER, if the terms of trade deterioration come from a rise of the imported goods price, the income and substitution effects act in opposite ways. The usually adopted hypothesis is that the substitution effect dominates and leads to a real exchange rate appreciation. Finally, the *EUV* variable takes the qualitative outcome of international specialisation improvement into account. The underlying idea is that the more important the export unit value is, the more able the country is to sustain a rise in its ERER level. On the contrary, if the country is specialised on standardised goods, it has to benefit from important price competitiveness and the ERER level has to be weak[19].
Given these proxies the expected long-run relationship is of the form :

$$RER_t = f\,(A_t, G_t, OPEN_t, TOT_t, EUV_t, GDP_t),$$

All the series are extracted from I.M.F International Financial Statistics and from World Bank World Table Data. The effective real exchange rates come from the J-P Morgan (1994) database that provides monthly data of RER for emergent countries. The data base covers the period 1979-1 / 1995-4 and all the empirical calculations have been undertaken with the PC-FIML software and CATS in RATS.
All variables are transformed in natural logarithm and in what follows lower-case letters denote the natural logarithm of the corresponding variable. The analysis first step is to look at the data univariate properties and determine their integratedness degree. We have therefore applied a sequence of standard unit root tests (Augmented dickey Fuller tests, namely Jobert's procedure (1992)), as well as Schmidt and Phillips' test (1992), Kwiatkowsky, Phillips and Shin test (KPSS) (1992)), to investigate which of the I(0), I(1), I(2) assumption is most likely to hold. The results of the Jobert procedure, of the Schmidt and Phillips' tests and of the KPSS tests, available from the authors upon request, indicate that all variables seem well characterised as an I(1) process, some with non-zero drift in the case of gdp and euv.

[19] For a discussion of the effect of international specialization on exchange rate level see for example Lafay (1981). More recently, Aglietta and *al*.(1997) introduced an indicator of price competitiveness in a stock /flow ERER determination approach.

5.2 Testing the number of cointegrating vectors and exogeneity hypothesis

Let us now consider a VAR-ECM model in which we investigate the presence of cointegrated vectors under the assumption that the selected variables are well represented by an I(1) process with potential deterministic component. Using sequences of Likelihood Ratio Tests, it turns out that the best specification of the deterministic components of the model consists of an unrestricted constant as well as a linear trend constrained to lie in the cointegrating space, thus ruling out quadratic trends. Table 2 presents the results of a ‘conventional’ cointegrating analysis using Johansen testing procedure, within a third order VAR-ECM.

TABLE 2 - Estimation of the number of cointegrating relationships

Mexico	λ_{max} test		λ_{trace} test	
Ho against Ha	Statistic	Critical value (at 5%)	Statistic	Critical value (at 5%)
r = 0 against r = 1	61.05 **	49.4	202.5 **	146.8
r ≤ 1 against r = 2	50.03 **	44.0	141.5 **	114.9
r ≤ 2 against r = 3	29.65	37.5	91.42 *	87.3
r ≤ 3 against r = 4	25.22	31.5	61.78	63.0
r ≤ 4 against r = 5	16.42	25.5	36.56	42.4
r ≤ 5 against r = 6	11.38	19.0	20.13	25.3
r ≤ 6 against r = 7	8.73	12.3	8.75	12.3

** and ** indicate respectively significance at 5 % and 10 % levels*

Employing a 5 % significance level, this suggests r = 2 according to the $\lambda_{\max}$ test and r = 3 according to the λ_{trace} test. Finally the cointegration graphical analysis leads to retain the hypothesis of the existence of two long-run relations between the variables[20].

It is important to notice at this stage that we are confronted here with the typical problem mentioned in section I, since two long run relationships have been detected empirically between real exchange rate and its fundamentals, whereas according to economic theory there should only exist one. In fact as our aim is to obtain an estimation of equilibrium real exchange for Mexico, we’d like to be able to focus exclusively on only one of these long-run relationship found previously and identify it as an equilibrium real exchange rate relationship. Furthermore, as our interest lies first of all in modelling real exchange rate and not in deriving empirical models for private absorption, public expenditures, openness degree of the

[20] This choice is also motivated by the fact that this third long-run relation is not significant at a 10 % level.

economy, terms of trade, exports unit value, real gdp per capita, a conditional approach could be well suitable if the remaining variables of the system were weakly exogenous for the parameters of the equilibrium real exchange rate relationship entering the real exchange rate equation and not for all long-run relationships. To this end the canonical decomposition and the weak exogeneity conditions respectively given in theorem 1 and proposition 1 can be of a great help.

Recalling that according to Edwards' theoretical model (1994), the endogenous and exogenous variables are respectively given by $Y_t^{'} =$ (rer_t) and $Z_t^{'} =$ (a_t, g_t, $open_t$, tot_t,euv_t, gdp_t), the identification of the two long run relationships is achieved by the following restrictions:

- we exclude euv from the first relationship and normalise the rer coefficient to one so as to identify this first cointegrating vector as an equilibrium real exchange rate relationship.
- We exclude rer from the second relationship and normalise the open coefficient to one[21] so as this second cointegrating vector can be interpreted as a purely exogenous long path.

We then test additional restrictions on some weights, namely of some specific parameters of the α matrix, related to the weak exogeneity hypothesis of $Z_t^{'}$ for the first long run relationships and its weights (see proposition 1). A Likelihood Ratio Test leads to accepting this hypothesis at the 1% level (χ^2 (7) = 17.83, p-value of 0.023). Thus this equilibrium real exchange rate relationship can be estimated without a loss of information from a single equation error-correction model (partial model), composed of the rer equation alone, ignoring the Z_t equations (marginal model), since all relevant information on this first long run relationship is contained in the partial system. Under these restrictions, we obtain the cointegrating and weight matrices reported in table 3.

[21] The OPEN and ERER coefficients have of course been checked to be different from zero before being normalised to one.

TABLE 3 - Just identified cointegrating vectors [22]

Variables	β matrix		α matrix	
rer	1.000	0.000	-0.151	0.000
a	2.329	0.990	0.000	-0.012
g	-1.400	2.842	0.000	0.077
open	0.412	1.000	0.000	-0.129
tot	1.560	-3.344	0.000	0.052
euv	0.000	3.710	0.000	0.015
gdp	3.146	5.153	0.000	0.047
trend	0.0251	-0.033		

It must be underlined that the usual weak exogeneity hypothesis for all long run parameters, which will lead to testing $\alpha_Z = 0$ in equation 3, is largely rejected by data at any level of significance (χ^2 (12) = 46.77, p-value of 0.000). This means we can't validly conduct inference on all cointegrating parameters from the rer equation error-correction model alone, which empirically clearly illustrates the fact that these usual weak exogeneity conditions are in our case too restrictive.
Moreover economically meaningful overidentifying restrictions can afterwards easily be tested on this equilibrium real exchange rate relationship, and one can of course also specify a structural error correction model, and identify the short run structure of this model following the approach advocated by Johansen and Juselius (1994), but it is here outside the scope of this paper.

Finally we test further restrictions (see proposition 2) related to strong exogeneity hypothesis from Y to Z (which combines weak exogenity hypothesis with non-causality hypothesis from Y to Z). A Likelihood Ratio Test leads to easily rejecting this hypothesis at any level of significance (χ^2 (9) = 99.451, p-value of 0.000), which means one cannot make valid forecast of rer from the single equation error-correction model alone, given forecasts of rer's determinants from the marginal model.

VII. Conclusion

In this paper, after having underlined both theoretically and empirically that the usual weak and strong exogeneity conditions reported in most of the papers of the existing literature may be in some cases too restrictive (if the investigator is only interested in a subset of cointegrated vectors that belongs to the equations describing the evolution of the endogenous

[22] Although our interest only lies in the equilibrium real exchange rate relationship (first column of the β matrix) and its weights, we report however the estimations of the full α and β matrices to clearly indicate the tested restrictions related to the weak-exogeneity hypothesis .

variables for instance), we have paid particular attention to proposing less restrictive necessary and sufficient exogeneity conditions. To this end we have first provided a framework based on two canonical representations of the long-run matrix Π in which the nullity of some parameters blocks implies no loss of generality, a first representation in the I (1) case and a more general second one, which can give rise to I (2) behaviour of the endogenous variables Y. These canonical decompositions are then shown to constitute a suitable basis in which less restrictive weak and strong exogeneity hypotheses can be tested and permit to clearly distinguish the nullity of some parameter blocks we can always achieve without any loss of generality using basis changes, of the nullity of the parameter blocks resulting from the tested hypothesis.

These exogeneity hypotheses have been given in the traditional VAR-ECM model in which we have made a distinction between endogenous and exogenous variables as well as in the VAR-ECM block recursive form and different sets of potential parameters of interest for investigators have been considered : ((i) one subset of cointegrating vectors involving both endogenous and exogenous variables, (ii) one subset of cointegrating vectors involving both endogenous and exogenous variables and another subset of purely exogenous long-run paths, (iii) short-run parameters combined with (i) or (ii)).

As a matter of fact these conditions are also theoretically appealing since they don't forbid the marginal model from including error correction mechanism anymore (and thus don't necessarily entail this model to be a VAR in first difference), since this model can now contain cointegrating relations that involve the Z_t variables alone. In addition in the more general canonical representation, the conditional and marginal models can even contain two completely separate sets of purely exogenous long run paths. These exogeneity conditions may also provide a solution to two classical related issues often encountered in empirical applications (as it is shown in the empirical illustration of section VI), namely (a) the existence of more cointegrating vectors between the variables under study, than stipulated by economic theory and (ii) the willingness to make inference on only a subset of cointegrating vectors (in accordance with economic theory) in a conditional (or partial) model. Finally all the proposed exogeneity conditions can be tested using asymptotically Khi square distributed Wald tests statistics and computed empirically using the PC-FIML software (1994, 1997) or CATS in RATS for instance, which make them very easy to investigate.

Appendix

Proof of theorem 1

The theorem is proved as follows :

First note that if we make a basis change in the cointegrating space such as $\beta^* = \beta$ P, where P is a non singular (r, r) matrix, then the α^* matrix is completely determined by :

$$\alpha\,\beta' = \alpha^*\beta'^* \;\Leftrightarrow (\alpha^* P' - \alpha)\,\beta' = 0$$
$$\Leftrightarrow \alpha^* P' - \alpha = 0, \text{ because } \beta' \text{ is of full rank column r}$$
$$\Leftrightarrow \alpha^* = \alpha\,(P')^{-1}$$

Next consider a β basis of the cointegrating space of rank r, EC = $\{v \in R^n : a_1\,\beta_1 + ... + a_r\,\beta_r = v\ .\ a_i \in R\}$, and let U_k be a vectorial subspace of R^n spanned by the set of vectors of the form $\begin{bmatrix} 0 \\ I_k \end{bmatrix}$. U_k intersection with the cointegrating space is also a vectorial subspace. This implies that a EC basis can be determined in completing a β_2 basis of EC $\cap$ U_k with β_1 vectors. We define r_1 as β_1 rank and r - r_1 as β_2 rank. Moreover since β_1 is also a supplementary space basis, it follows that β_{Y1} is of full rank column r_1 : in other words, if a linear combination of the β_{Y1} columns that produces a column of zeros existed, this would mean that a vector of U_k spanned by the β_1 vectors would exist, which has been excluded by construction.

It is easily shown that the transformation matrix from the β basis to the $\beta^* = \beta$ P basis can be written as

$$P = \left[\begin{array}{c|c} I_{r_1} & -C \\ \hline 0 & I_{r-r_1} \end{array}\right]$$ and then one deduces $\alpha^* = \alpha\,(P')^{-1}$.

Finally the expressions of the reparametrised matrices β and α are given by

$$\beta = \left[\beta_1 \;\|\; \beta_2\right] = \left[\begin{array}{c|c} \beta_{Y1} & 0 \\ \hline \beta_{Z1} & \beta_{Z2} \end{array}\right], \quad \alpha = \left[\alpha_1 \;\|\; \alpha_2\right] = \left[\begin{array}{c|c} \alpha_{Y1} & \alpha_{Y2} \\ \hline \alpha_{Z1} & \alpha_{Z2} \end{array}\right],$$
$$\underset{r_1}{\longleftrightarrow}\quad \underset{r-r_1}{\longleftrightarrow} \qquad\qquad \underset{r_1}{\longleftrightarrow}\quad \underset{r-r_1}{\longleftrightarrow}$$

with $\underset{(n,r_1)}{\beta_1} = \overset{(g,r_1)}{\underset{(k,r_1)}{\left[\frac{\beta_{Y1}}{\beta_{Z1}}\right]}}$, $\underset{(n,r-r_1)}{\beta_2} = \overset{(g,r-r_1)}{\underset{(k,r-r_1)}{\left[\frac{0}{\beta_{Z2}}\right]}}$, $\underset{(n,r_1)}{\alpha_1} = \overset{(g,r_1)}{\underset{(k,r_1)}{\left[\frac{\alpha_{Y1}}{\alpha_{Z1}}\right]}}$, $\underset{(n,r-r_1)}{\alpha_2} = \overset{(g,r-r_1)}{\underset{(k,r-r_1)}{\left[\frac{\alpha_{Y2}}{\alpha_{Z2}}\right]}}$.

This completes the proof of Theorem 1.

♦

Proof of theorem 2

(i) The first result given in the theorem is proved as follows :

First note that any basis change in the dual space of the cointegrating space of the form $\alpha^* = \alpha Q$, where Q is a non singular (r, r) matrix, uniquely determines the β^* matrix, whose expression is given by $\beta^* = \beta (Q^{-1})'$.

Next, given the expressions of the α and β matrices reported in theorem 1, we make a new reparametrisation, but this time in the dual space of the EC cointegrating vectors space, namely in the α adjustment space. To this end, we now consider Im α_2 [23]and we make a new basis change in order to have a Im $\alpha_2 \cap$ Im $\begin{bmatrix} 0 \\ I_k \end{bmatrix}$ basis. It is then easily shown that the transformation matrix from the α^* basis to the α^{**} basis can be written as :

$$Q = \left[\begin{array}{c|c|c} I_{r_1} & 0 & 0 \\ \hline 0 & I_{r^*} & B' \\ \hline 0 & 0 & I_{r_2} \end{array}\right]$$, which leads to a new α of the form :

$$\alpha = \left[\alpha_1 \;\|\; \alpha_{21} \;\|\; \alpha_{22}\right] = \left[\begin{array}{c|c|c} \alpha_{Y1} & \alpha_{Y21} & \alpha_{Y22} \\ \hline \alpha_{Z1} & 0 & \alpha_{Z22} \end{array}\right].$$

$$\qquad \overleftrightarrow{\;r_1\;} \quad \overleftrightarrow{\;r^*\;} \quad \overleftrightarrow{\;r_2\;}$$

Finally the corresponding β given by the expression $\beta^{**} = \beta^* (Q^{-1})'$ can easily be checked to be the one reported in theorem 2 :

[23] Im α_2 represents the α_2 image space.

$$\beta = \left[\beta_1 \;\middle\|\; \beta_{21} \;\middle\|\; \beta_{22}\right] = \left[\frac{\beta_{Y1}}{\beta_{Z1}} \;\middle\|\; \frac{0}{\beta_{Z21}} \;\middle\|\; \frac{0}{\beta_{Z22}}\right],$$

$$\underset{r_1}{\longleftrightarrow} \quad \underset{r^*}{\longleftrightarrow} \quad \underset{r_2}{\longleftrightarrow}$$

♦

(ii) The second result given in theorem 2 is proved as follows :

First of all, let us consider a basis change in the cointegrating space of the form $\beta^* = \beta$ M, which entails $\alpha^* = \alpha\ (M')^{-1}$ where M is a non singular (r, r) matrix and let us determine the conditions that must hold on this transformation matrix M to have $\begin{cases} rank\,(\beta^*_{Y1}) = rank\,(\beta_{Y1}) = r_1 \\ \beta^*_{Y2} = \beta_{Y2} = 0 \end{cases}$.

Partitioning M into $\left[\frac{M_{11}}{M_{21}} \;\middle\|\; \frac{M_{12}}{M_{22}}\right]$ enables us to write :

$$\beta^* = \left[\frac{\beta^*_{Y1}}{\beta^*_{Z1}} \;\middle\|\; \frac{0}{\beta^*_{Z2}}\right] = \left[\frac{\beta_{Y1}}{\beta_{Z1}} \;\middle\|\; \frac{0}{\beta_{Z2}}\right] \left[\frac{M_{11}}{M_{21}} \;\middle\|\; \frac{M_{12}}{M_{22}}\right] = \left[\frac{\beta_{Y1}\, M_{11}}{\beta_{Z1}\, M_{11} + \beta_{Z2}\, M_{21}} \;\middle\|\; \frac{\beta_{Y1}\, M_{12}}{\beta_{Z1}\, M_{12} + \beta_{Z2}\, M_{22}}\right]$$

$\beta_{Y1}\, M_{12} = 0 \Leftrightarrow M_{12} = 0$, rank ($\beta_{Y1}\, M_{11}$) = $r_1 \Leftrightarrow M_{11}$ inversible and M inversible implies M_{22} inversible. This means that the conditions (a) $M_{12} = 0$ as well as (b) M_{11} and M_{12} inversible must hold on this transformation matrix M for it to be invariant, namely to have to have $\begin{cases} rank\,(\beta^*_{Y1}) = rank\,(\beta_{Y1}) = r_1 \\ \beta^*_{Y2} = \beta_{Y2} = 0 \end{cases}$

Next, the corresponding α given by $\alpha^* = \alpha\ (M')^{-1}$ can easily be checked to have the following expression :

$$\alpha^* = \left[\frac{\alpha_{Y1}}{\alpha_{Z1}} \;\middle\|\; \frac{\alpha_{Y2}}{\alpha_{Z2}}\right] \left[\frac{M'^{-1}_{11}}{0} \;\middle\|\; \frac{-M'^{-1}_{11} M'_{21} M'^{-1}_{22}}{M'^{-1}_{22}}\right] = \left[\frac{\alpha_{Y1}\, M'^{-1}_{11}}{\alpha_{Z1}\, M'^{-1}_{11}} \;\middle\|\; \frac{-\alpha_{Y1}\, M'^{-1}_{11} M'_{21} M'^{-1}_{22} + \alpha_{Y2}\, M'^{-1}_{22}}{-\alpha_{Z1}\, M'^{-1}_{11} M'_{21} M'^{-1}_{22} + \alpha_{Z2}\, M'^{-1}_{22}}\right],$$

namely

$$\alpha^* = \left[\frac{\alpha^*_{Y1}}{\alpha^*_{Z1}} \middle\| \frac{\alpha^*_{Y2}}{\alpha^*_{Z2}}\right].$$

If $\alpha_{Z1} = 0$, then $\alpha^*_{Z2} = \alpha_{Z2} M^{'-1}_{22}$ and we have rank (α^*_{Z2}) = rank (α_{Z2}) = r_2, which means that r_2 is uniquely defined and is invariant to the chosen reparametrisation.

♦

Proof of proposition 1

The parameters of the conditional and marginal models are respectively

$$\varphi^+_Y = (\Gamma^+_{YY,i} \quad i=1,\dots,p-1, \quad \Gamma^+_{YZ,i} \quad i=0,\dots,p-1 \quad ,\alpha^+_{Y1},\left[\beta'_{Y1} \middle\| \beta'_{Z1}\right] \alpha^+_{Y1}, \beta'_{Z1},\alpha^+_{Y2}, \beta'_{Z2}, \Sigma^+_{YY})$$

and

$$\varphi_Z = (\Gamma_{ZY,i} \quad i=1,\dots,p-1, \quad \Gamma_{ZZ,i} \quad i=1,\dots,p-1 \quad \alpha_{Z1},\left[\beta'_{Y1} \middle\| \beta'_{Z1}\right] \alpha_{Z1},\beta'_{Z1},\alpha_{Z2},\beta'_{Z2}, \Sigma_{ZZ}).$$

Comparing these two sets of parameters we can see that the parameters of the conditional and marginal models are not variation free since β'_{Y1}, β'_{Z1}, β'_{Z2} appear both in these two models.

Let us consider now the case when $\begin{cases} \alpha_{Z1} = 0 \\ \alpha^+_{Y2} = 0 \end{cases}$. From the multivariate normal distribution, we know that the parameters (Σ_{ZZ}, Σ_{YZ}, Σ^{-1}_{ZZ}, Σ^+_{YY}) are variation free (see Barndorff-Nielsen [1978]); thus it is also the case for the parameters φ^+_Y et φ_Z and the sequential cut definition is satisfied. Furthermore the parameters of interest are only function of those of the conditional model, which completes the proof of proposition 1.

♦

Proof of corollary 1

The parameters of the conditional and marginal models are respectively

$$\varphi^+_Y = (\Gamma^+_{YY,i} \quad i=1,\dots,p-1, \quad \Gamma^+_{YZ,i} \quad i=0,\dots,p-1 \quad ,\alpha^+_{Y1},\left[\beta'_{Y1} \middle\| \beta'_{Z1}\right] \alpha^+_{Y1}, \beta'_{Z1},\alpha^+_{Y2}, \beta'_{Z2}, \Sigma^+_{YY})$$

and

$$\varphi_Z = (\Gamma_{ZY,i} \quad i=1,\ldots,p-1, \quad \Gamma_{ZZ,i} \quad i=1,\ldots,p-1 \quad \alpha_{Z1}, \left[\beta'_{Y1} \,\|\, \beta'_{Z1}\right], \alpha_{Z1}, \beta'_{Z1}, \alpha_{Z2}, \beta'_{Z2}, \Sigma_{ZZ}).$$

Comparing these two sets of parameters we can see that the parameters of the conditional and marginal models are not variation free since β'_{Y1}, β'_{Z1}, β'_{Z2} appear both in these two models.

Consider now the case when $\begin{cases} \alpha_{Z1} = 0 \\ \alpha_{Y2} = 0 \\ \Sigma_{YZ} = 0 \end{cases}$ which entails $\begin{cases} \alpha_{Z1} = 0 \\ \alpha^{+}_{Y2} = 0 \end{cases}$. From the multivariate normal distribution, we know that the parameters (Σ_{ZZ}, Σ_{YZ}, Σ^{-1}_{ZZ}, Σ^{+}_{YY}) are variation free (see Barndorff-Nielsen [1978]); thus it is also the case for the parameters φ^{+}_{Y} et φ_Z and the sequential cut definition is satisfied.

Furthermore the parameters of interest are only function of those of the conditional model, which completes the proof of corollary 1.

♦

Proof of proposition 2

As it is now well known (see Toda and Philips (1991, 1993 for instance), Granger short-run non-causality hypothesis from Y to Z is equivalent to $\Gamma_{ZY,i} = 0, i = 1,\ldots p-1$ and the only additional hypothesis that must be satisfied here is Granger long-run non-causality from Y to Z. This hypothesis is verified if and only if $\alpha_{Z1}\beta'_{Y1} = 0$ which necessary entails $\alpha_{Z1} = 0$, since β'_{Y1} is of full rank row r_1.

♦

Proof of proposition 3

The only thing to be proved in addition to proposition 1 is Granger non-causality hypothesis from Y to Z. We know that long-run non-causality hypothesis from Y to Z ($\alpha_{Z1} = 0$) is already verified because of the sequential cut required by weak exogeneity. If now

$\Gamma_{ZY,i} = 0, i = 1,...p-1$, then short-run non-causality hypothesis from Y to Z is also verified, which completes the proof of proposition 3.

♦

Proof of proposition 4

Z_t is weakly exogenous for the parameters of the conditional model, if and only if these parameters can be calculated from the conditional model alone (equation 4’’) and if the parameters of respectively the conditional and marginal models (φ^+_Y and φ_Z) are not subject to cross restrictions and are variation free (sequential cut).

• Let us prove in a first stage that weak exogeneity of Z_t for the parameters of equation 4’’necessarily implies $\begin{cases} \alpha_{Z1} = 0 \\ \alpha^+_{Y22} = 0 \end{cases}$.

To this end we must compare equations 4’ and 5’ to see the parameters that are common to these equations. First it can easily be seen that β'_{Y1} both appears in the conditional and marginal models through the expressions $\alpha^+_{Y1}\beta'_{Y1}$ and $\alpha_{Z1}\beta'_{Y1}$. Since α^+_{Y1} and β'_{Y1} are respectively of full rank column and row r_1, the term $\alpha^+_{Y1}\beta'_{Y1}$ can’t be equal to zero and hence the term $\alpha_{Z1}\beta'_{Y1}$ must obligatory do. Furthermore we know that $\alpha_{Z1}\beta'_{Y1} = 0$ if and only if $\alpha_{Z1} = 0$, because β'_{Y1} is of full rank row r_1.

Then the only parameter in common to the two models is β_{Z2} (respectively $\alpha^+_{Y2}\beta'_{Z2}$ and $\alpha_{Z2}\beta'_{Z2}$ in the conditional and marginal model). The term $\alpha_{Z2}\beta'_{Z2}$ can’t vanish since on the one side α_{Z2} is of full rank column and on the other β'_{Z2} is different from zero by construction. Consequently it is the term $\alpha^+_{Y2}\beta'_{Z2}$ that must disappear. The only way for this to happen is when $\alpha^+_{Y2} = 0$.

This proves that $\begin{cases} \alpha_{Z1} = 0 \\ \alpha^+_{Y22} = 0 \end{cases}$ are necessary conditions.

• Let us prove in a second stage that if $\begin{cases} \alpha_{Z1} = 0 \\ \alpha^{+}_{Y22} = 0 \end{cases}$, then Z_t is weakly exogeneous for the parameters of equation 4''

The parameters of the conditional and marginal models are respectively

$\varphi^{+}_{Y} = (\Gamma^{+}_{YY,i} \quad i=1,\ldots,p-1, \quad \Gamma^{+}_{YZ,i} \quad i=0,\ldots,p-1 \quad, \alpha^{+}_{Y1}, \left[\beta'_{Y1} \| \quad \beta'_{Z1}\right], \alpha^{+}_{Y21}, \beta'_{Z21}, \alpha^{+}_{Y22}, \beta'_{Z22}, \Sigma^{+}_{YY})$

and

$\varphi_{Z} = (\Gamma_{ZY,i} \quad i=1,\ldots,p-1, \quad \Gamma_{ZZ,i} \quad i=1,\ldots,p-1 \quad, \alpha_{Z1}, \left[\beta'_{Y1} \| \quad \beta'_{Z1}\right] \alpha_{Z22}, \beta'_{Z22}, \Sigma_{ZZ})$.

Comparing these two sets of parameters we can see that the parameters of the conditional and marginal models are not variation free since β'_{Y1}, β'_{Z1}, β'_{Z22} appear both in these two models.

Consider now the case when $\begin{cases} \alpha_{Z1} = 0 \\ \alpha^{+}_{Y22} = 0 \end{cases}$. From the multivariate normal distribution, we know that the parameters (Σ_{ZZ}, Σ_{YZ}, Σ^{-1}_{ZZ}, Σ^{+}_{YY}) are variation free (see Barndorff-Nielsen [1978]); thus it is also the case for the parameters φ^{+}_{Y} et φ_{Z} and the sequential cut definition is satisfied. Furthermore the parameters of interest are only function of those of the conditional model, which completes the proof of proposition 4.

♦

Proof of corollary 3

• Let us prove in a first stage that weak exogeneity of Z_t for the parameters of equation 2''necessarily implies $\begin{cases} \alpha_{Z1} = 0 \\ \alpha_{Y22} = 0 \\ \Sigma_{YZ} = 0 \end{cases}$.

From proposition 4 the result is straightforward since $\begin{cases} \alpha_{Z1} = 0 \\ \alpha^{+}_{Y22} = 0 \end{cases} \Leftrightarrow$ $\begin{cases} \alpha_{Z1} = 0 \\ \alpha_{Y22} - \Sigma_{YZ}\Sigma^{-1}_{ZZ}\alpha_{Z22} = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_{Z1} = 0 \\ \alpha_{Y22} = 0 \\ \Sigma_{YZ} = 0 \end{cases}$, since the Σ_{ZZ} matrix is invertible and because α_{Z22} can't vanish, the α matrix being of full rank column r.

• Let us prove in a second stage that if $\begin{cases} \alpha_{Z1} = 0 \\ \alpha_{Y22} = 0 \\ \Sigma_{YZ} = 0 \end{cases}$, then Z_t is weakly exogeneous for the parameters of equation 4''

The parameters of the conditional and marginal models are respectively

$\varphi^{+}_{Y} = (\Gamma^{+}_{YY,i} \quad i=1,...,p-1, \quad \Gamma^{+}_{YZ,i} \quad i=0,...,p-1 \quad ,\alpha^{+}_{Y1},\left[\beta'_{Y1} \,\Vert\, \beta'_{Z1}\right],\alpha^{+}_{Y21},\beta'_{Z21},\alpha^{+}_{Y22},\beta'_{Z22},\Sigma^{+}_{YY})$

and

$\varphi_{Z} = (\Gamma_{ZY,i} \quad i=1,...,p-1, \quad \Gamma_{ZZ,i} \quad i=1,...,p-1 \quad ,\alpha_{Z1},\left[\beta'_{Y1} \,\Vert\, \beta'_{Z1}\right]\alpha_{Z22},\beta'_{Z22},\Sigma_{ZZ})$.

Comparing these two sets of parameters we can see that the parameters of the conditional and marginal models are not variation free since β'_{Y1}, β'_{Z1}, β'_{Z22} appear both in these two models.

Consider now the case when $\begin{cases} \alpha_{Z1} = 0 \\ \alpha_{Y22} = 0 \\ \Sigma_{YZ} = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_{Z1} = 0 \\ \alpha^{+}_{Y22} = 0 \end{cases}$. From the multivariate normal distribution, we know that the parameters (Σ_{ZZ}, Σ_{YZ}, Σ^{-1}_{ZZ}, Σ^{+}_{YY}) are variation free (see Barndorff-Nielsen [1978]); thus it is also the case for the parameters φ^{+}_{Y} et φ_{Z} and the sequential cut definition is satisfied. Furthermore the parameters of interest are only function of those of the conditional model, which completes the proof of corollary 3.

♦

Proof of proposition 5

The only thing to be proved in addition to proposition 1 is Granger non-causality hypothesis from Y to Z. We know that long-run non-causality hypothesis from Y to Z (α_{Z1} = 0) is already verified because of the sequential cut required by weak exogeneity. Thus there only remains to establish the conditions under which short-run non-causality hypothesis from Y to Z holds, which is straightforward, since short-run non-causality hypothesis from Y to Z is well known to be equivalent to $\Gamma_{ZY,i} = 0, i = 1,...p-1$ (see Toda and Phillips (1991, 1993) for instance) .

♦

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