

# FINANCIAL MARKETS, CREATION AND LIQUIDATION OF FIRMS AND AGGREGATE DYNAMICS

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## Abstract

This paper models the impact of interest rate shocks on the creation and destruction of firms. The mechanism by which these shocks are transmitted across firms behavior is based on limited commitment to contractual arrangements. A firm's temptation to default limits the provision of credit by the bank. Credit constraints arise endogenously as part of the optimal long-term relationship between banks and firms. The history of the relationship endogenously determines the credit limits. In this environment, a positive interest rate shock decreases the survival probability of constrained firms. In contrast, the unconstrained firms respond primarily by adjusting their scale of operations. In the aggregate, increases in the interest rate force some of the existing firms into liquidation and reduce the number of newly created ones. We study the implied behavior of the series of creation and liquidation of jobs and firms. Then, we use the model to discuss the implied response of aggregate output and the role played by fluctuations in the cross-section of firms.

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# 1 Introduction

This paper develops an equilibrium model in which interest rate shocks lead to fluctuations in the creation and destruction of firms. It examines economies in which limited commitment restricts the operations of firms by endogenously imposing limits on their credit. The potential default of entrepreneurs constraints the optimal dynamic contract between banks and firms. Alternatively, limited commitment imposes limits to the debt and credit in decentralized securities markets. The economies are populated by firms in many different stages of their life cycle and with many different sizes. The model provides an explicit link between age, size and the credit constraints of the firms as well as the effect of shocks on their entry, exit and growth. The aggregate response of the economy to interest rates fluctuations is determined by its effect on the cross section of active firms.

New and old firms respond to interest rate shocks depending on their history. Over their life cycle, firms accumulate collateral. When young, firms are likely to have little collateral and therefore to be constrained and small. They respond to movements in interest rates primarily through their decision to continue producing. In contrast, older firms are more likely to be unconstrained and they respond to interest rates movements by adjusting their scale of operations. During recessions led by high interest rates, more young firms will be liquidated while old firms will reduce their scale of operation.

The model predicts that the destruction of firms is concentrated in the first periods of increasing interest rates. After a burst of destruction, surviving firms are more likely to remain active, even if interest rates remain high. The model assumes frictions in the creation rate of firms and endogenously generates frictions in the growth and survival of each firm. Both forces add persistence to the response of output to interest rate shocks. Moreover, aggregate frictions in the rate of creation imply that recessions are shorter and sharper than recoveries. We explore whether the non-trivial life cycle dynamics by itself adds to this feature of aggregate output dynamics.

Shocks and frictions in credit markets have received wide attention for aggregate dynamics and for interpreting the observed behavior of individual firms. Recent papers by T. Dunne, M. Roberts and L.Samuelson [24, 25], D. Evans [28, 29] and B. Hall [33] conclude that young, small firms grow faster but die more frequently than older, larger ones. At a more aggregate level some authors find that smaller firms (M. Gertler and S. Gilchrist [31]) or firms with less access to financial markets( A. Kashyap, J. Stein and O. Lamont, and D. Wilcox [42, 40]) are more responsive to monetary shocks than their larger or less constrained counterparts. Also, R. Hall [33] and S.Davis and J.Haltiwanger [18] report an association between the flows of destruction

of jobs and liquidation of plants with real interest rates.

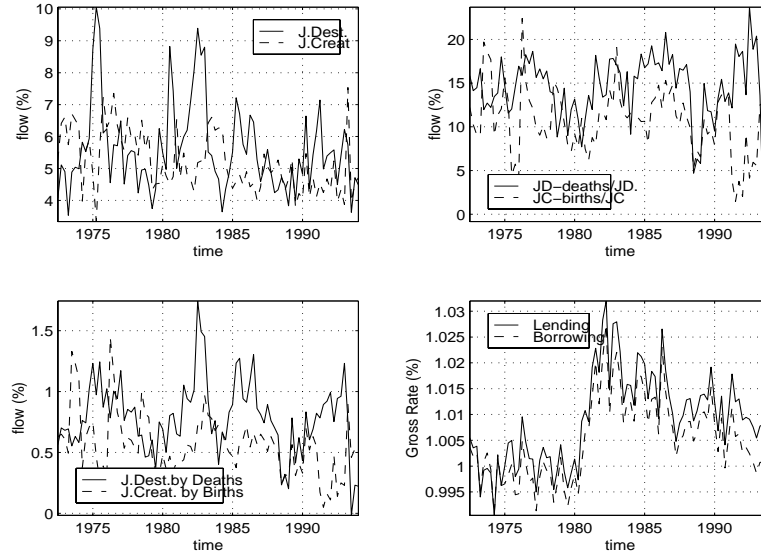
In the context of the model, the limited contract enforceability provides the link between the firm age and size with its access to external financing in an environment with shocks to the interest rates as the only source of aggregate fluctuations. It emphasizes the liquidation and creation of firms in the transmission of interest rate shocks. Therefore, the model has observable implications at both micro and macro levels. Yet, the existent empirical work runs short with respect to the predictions of the model. The empirical work cited above on firm dynamics uses data recollected every five years, and therefore cannot be linked with business cycle fluctuations, and even less with interest rates or other indicators of financial markets. The cited work on asymmetric responses of small and large firm between responses in the extensive margins (entry, exit) from responses in the intensive margin. Specifically, [31] ignore the composition and use the aggregate of firm groups while [42] eliminate from the sample the firms liquidated during the period.

Because the interest rate transmission mechanism works through credit constraints, we derive these constraints endogenously from features of the environment. The model is closely related to recent literature on limited enforcement on dynamic economies. Specifically, we borrow intensively from the seminal work of R. Albuquerque and H. Hopenhayn [1] who in turn extend the previous work by O.Hart and J.Moore [35]. Albuquerque and Hopenhayn find that their model of dynamic credit constraints can explain the observed patterns of firm dynamics.

Our focus in this paper, is on the behavior of aggregate creation and destruction flows. We extend the work of Albuquerque and Hopenhayn by including fluctuations in the interest rate in their contract design problem. Besides other relevant results, we extend the work of Albuquerque and Hopenhayn by providing a decentralization scheme in which entrepreneurs trading in one period securities can replicate the allocations from the optimal infinite horizon contract. This result, similar to the one in F.Alvarez and U.Jermann[2], hinges on solvency and borrowing constraints. The decentralization scheme makes it natural to link the implied asymmetries of interest rates to the collateral or net worth of the entrepreneur.

The role of financial markets for aggregate fluctuations has traditionally been a center of attention. Recently, Bernanke-Gertler [5] and Fisher [30] study how incentive constraints propagate shocks by rationing the credit of a subset of firms. By design, their models cannot address the responses on the margins of creation and destruction of firms. T. Cooley and V. Quadrini [16] construct a model suitable to study these margins. In contrast to Cooley and Quadrini, we do not impose the form of contracts, but we derive the *optimal infinite horizon contracts* between banks and firms, and then derive the age and state contingent constraints required to achieve the allocations in decentralized one-period securities. Moreover, they focus on parameter

Figure 1: U.S.: Realization of Gross Flows and Interest Rates



configuration in which interest rates do not affect entry or exit. The attention here is on the response of series of activation and liquidation of firms.

The rest of the paper proceeds as follows: In the next section, we briefly overview the actual time series for the U.S. economy. In the third section we lay down the environment. The fourth section characterizes the optimal, infinite horizon contract between a bank and entrepreneur and discuss the implied firm dynamics, including entry and exit decisions. With the optimal contract at hand, in the fifth section we construct the solvency and borrowing constraints so that entrepreneurs trading in one-period securities can replicate the allocations of the infinite horizon relationships. The sixth section discusses the notions of credit constraints in the model, concluding that the least productive firms are the most constrained. In the seventh we examine more closely the asymmetric effects of interest rate shocks on the entry and exit decisions. The eighth section examines the implied aggregate dynamics, and provides a numerical illustration, discussing the limitations of the environment. The last section contains our conclusions and discusses extensions. The appendix has three sections. The first contains the proofs, the second provides a general equilibrium interpretation and the third describes the computational algorithm.

## 2 Interest Rates, Gross Flows and Output in the U.S.

Before getting to the details of the model, we examine the main qualitative predictions of the model in the data: First, the rate of destroyed firms is positively correlated and the rate of newly created firms is negatively correlated with interest rate shocks; moreover, the survival of small and young firms are the most sensitive. Second, the response of firm destruction to shocks must be highly concentrated in short periods of time, as surviving firms are likely to endure higher interest rates, and indeed, lower interest rate may induce higher future destruction flows. Finally, aggregate output responds negatively to interest rate shocks, not only from the reduced mass of active firms but also for the reduction in the production of remaining ones. Frictions in the creation rate of new firms and on the expansion of the existent firms generates persistence and asymmetry in the aggregate response.

The empirical literature on gross job flows (see Davis, Haltiwanger and Schuh [20] and the references therein) highlights a set of stylized facts: Gross flows are large, even in periods of macro stability. Job destruction is countercyclical while job creation is procyclical. Most notably, job destruction is more volatile than job creation. These facts are evident from a casual inspection of Figure 1.<sup>1</sup> As much as 10% of manufacturing jobs were destroyed in a single quarter, 1975:I; in most quarters around 5% of the jobs are newly created or destroyed. Clockwise, the top-left panel shows the rate of job creation and destruction for the entire manufacturing sector. The second shows the job creation and job destruction accrued by plant start-ups and plant shut-downs respectively. The third panel shows the fraction of total job creation and total job destruction that are accounted for by, respectively, start-ups and liquidations. The bottom-right panel exhibits the gross *real* borrowing (FedFunds) and lending (Prime) interest rates.<sup>2</sup>

Our model emphasizes the rate of activation and liquidation of firms. Unfortunately, there is no high frequency data available on these series. Following J.Campbell [12], however, we can interpret the job destruction (creation) from plant liquidation (startup) as a *employment-weighted* exit (entry) exit rates.<sup>3</sup> While the rate of jobs destroyed or created by exit or entry is not high, it can account for as much as 24% of the total manufacturing flow. Furthermore, the

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<sup>1</sup>The data runs for the period of 1972:II to 1993:IV was taken from Davis-Haltiwanger-Schuh.

<sup>2</sup>The inflation rate was calculated from the CPI. We take geometric average the three annualized gross nominal return implied by the interest rate for each of the month of the quarter and divided it by the annualized geometric average of the *effective* monthly gross inflation rate. In quarterly averages, the two rates comove very closely, and the results discussed below are insensitive to choosing either series.

<sup>3</sup>This is the best series available at business cycle frequencies as confidentiality clauses impede a direct access to the Census data.

Figure 2: U.S.: Correlation of Gross Flows Levels with Leads/Lags of Real Interest Rates

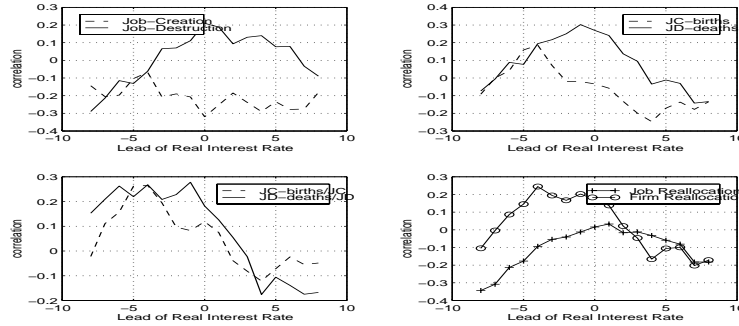
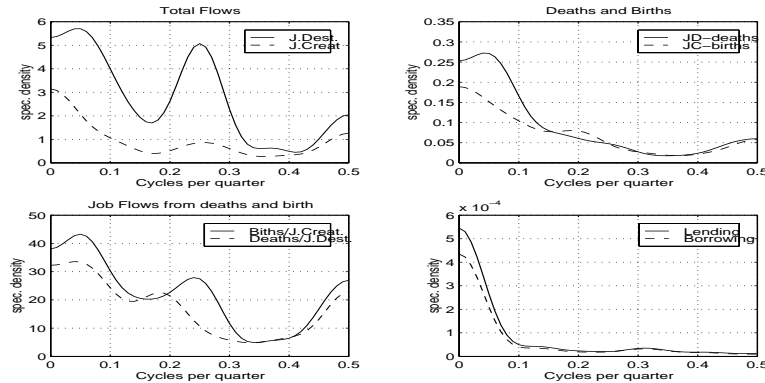


Figure 3: U.S.: Spectra of Gross Flows and Real Interest Rates



quantitative relevance of entry and exit is likely to be underestimated.<sup>4</sup> Observe that destruction flows has many spikes and many of those coincide with episodes of increasing interest rates.

Indeed, looking at Figure 2, it is clear that the gross flows comove with leads and lags of the real interest rate.<sup>5</sup> Job destruction (creation) is positively (negatively) correlated with real interest rates. The correlations with destruction series is higher than with creation series. Shutdowns correlate more with interest rates than total job destruction. Moreover, the correlations for the destruction (and reallocation) series have an inverted U-shape that peaks at the zero lag/lead. In general the correlations are not too high, but it is interesting to observe that the correlations of interest rates are not too far from the obvious rival, total factor productivity innovations, as reported by Campbell[12].

<sup>4</sup>Davis-Haltiwanger-Schuh [20], explain that their methodology is likely to reduce the participation of shut downs and start ups. Moreover, entering plants may intensively use temporary workers and delay recording them, and moreover effectively shut plants may retain a few employees just for tax, maintenance or other purposes not related with the activities of the plant.

<sup>5</sup>Series were seasonally adjusted using OLS on quarter dummies.

Figure 4: U.S.: Correlation of Net Flows Levels with Leads/Lags of Real Interest Rates

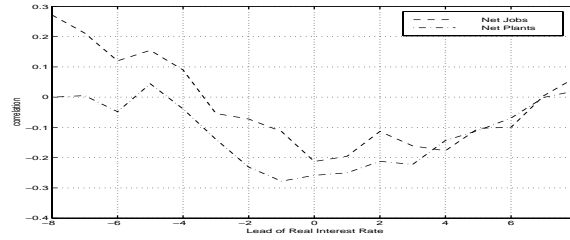
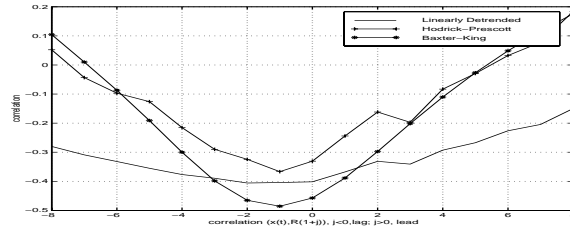


Figure 5: U.S.: Correlation of Output with Leads/Lags of Real Interest Rates



In Figure 1. we saw that there are short episodes of time in which the destruction flows are unusually high. This observation has led to view the destruction flows as *temporally concentrated*. One way to examine the degree of temporal concentration of the series, is by assessing the importance of fast reverting cycles –high frequencies components– in the series. Inspecting the spectra of the series, shown in Figure 5., also provides a comprehensive and parsimonious summary of the stochastic properties of the series, that will help us specifying the features of the data for which our model is relevant.

The spectra of all the destruction series is almost uniformly above of the corresponding creation series.<sup>6</sup> Seasonal frequencies (cycles of 4 quarters) are important, specially for the destruction series.<sup>7</sup> The low frequencies are a very important part of the series, indicating the relevance of the persistent decline of manufacturing during the sample period. The secular trend and the seasonal variations present in the manufacturing data overshadows the temporal concentration of the series. Yet, the spikes observed in the time domain manage to reflect themselves in a rising tail at high frequencies.<sup>8</sup>

Figure 3. shows a negative relation of interest rates with the net employment growth (creation-destruction) and net establishment growth (births-deaths). In fact, the net establishment growth correlations are stronger. The effect of interest rates on employment and number

<sup>6</sup>Which is consistent with the relative volatilities, as the area of the spectral density equals the estimated variance of the series.

<sup>7</sup>Of course, here the series *are not* seasonally adjusted.

<sup>8</sup>It would be interesting to observe data from the whole economy or sectors without secular trends.

of plants are reflected in negative relations with aggregate output as shown in Figure 4. The results are obtained by filtering the log of real aggregate output and interest rates with the Hodrick-Prescott, Baxter-King, and linear detrending filters.<sup>9</sup> Indeed, as highlighted by a variety of authors, e.g. King-Watson, [43], real interest rates are a negative lead indicator of aggregate output.

### 3 The Environment

#### Demographics and Preferences:

Time is discrete and the horizon is infinite; periods are indexed by  $t = \dots, -1, 0, 1, 2, 3, \dots$ . Each period brings along a vintage of many new *potential entrepreneurs*. All vintages have mass  $\delta$ . I shall refer to potential entrepreneurs as *agents*. In the beginning of every period a fraction  $\delta \in (0, 1)$  of the agents dies. The death probability of each agent is  $\delta$ . The total population in the economy is always equal to one. Agents are risk neutral and discount the future geometrically by a factor  $\beta \in (0, 1)$ . Thus, their evaluation of own consumption processes  $\{c_t\}$  as of any time  $t_o$  is given by

$$E \left\{ \sum_{t \geq t_o} (\beta(1 - \delta))^t c_t \mid F_{t_o} \right\}$$

where the probability of death renders an effective discount factor of  $\beta(1 - \delta)$ , and  $F_{t_o}$  denotes all the information available as of  $t_o$ .

#### Technologies:

Entrepreneurs are born with zero endowments.<sup>10</sup> Upon birth, each entrepreneur has access to two *mutually exclusive* technologies, stochastically identical across agents. The first is a *productive technology*. Its activation requires a set-up investment cost  $K_o > 0$ . In each period of activity, it also requires resources (working capital) to produce output. In actual economies, working capital takes the form of materials, inventories of finished goods, hours of labor, etc.

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<sup>9</sup>For Baxter-King we use (6,32,12), their preferred parameterization for quarterly series, i.e. remove cycles that last less than 6 quarters and more than 8 years, using 12 leads and lags in the moving average. For Hodrick-Prescott we set  $\lambda = 1600$

<sup>10</sup>The results of the paper are robust to assuming that the entrepreneurs are born with a small endowment. However, if the initial endowment is large enough, the incentive problems would disappear, as a bond-posting scheme will be available (on the long term relationships) or they do not need to borrow at all (in the decentralized setting).

Here I bundle all components into a single scalar variable  $k$ . With an amount of  $k$ , the technology current production is  $zf(k)$  where  $\{z^i\}$  is a stochastic process of productivities idiosyncratic to firm  $i$ , and  $f(\cdot)$  is a non-negative, strictly increasing, concave function. I will assume:

**A. 1 (F).**  $f(k) = k^\alpha$ ,  $\alpha \in (0, 1)$ ,

**A. 2 (P).**  $\{z^i\}$  is a stationary, ergodic, process, that is identical and independently distributed across agents of all vintages. It has finite support  $Z = \{z_1, z_2, \dots, z_n\}$ ,  $n < \infty$ . The transitions are given by  $P_z(\cdot, z)$ , and the unique invariant c.d.f. by  $F_z(\cdot)$ . The processes  $\{z^i\}$  have **positive persistence** over time, i.e. if  $z_0 < z_1 \in Z$ , then,  $P_z(\cdot, z_1) \geq_f P_z(\cdot, z_0)$ , where  $\geq_f$  denotes first order stochastic dominance.

For simplicity, I assume the option of activating the productive technology is available *only* in the first period of life for each agent, and it can be operated as long as the agent has not abandoned it or the firm has been liquidated.

The second option is a "backyard" technology. It is always available to the agent. We also use the term *underground* for this option, which requires no set up costs and produces a constant flow  $e > 0$  each period.<sup>11</sup> Once the agent has opted for the underground his utility  $U$  is simply

$$U = \frac{e}{1 - \beta(1 - \delta)}$$

In what follows, agents operating the productive technology are called *active entrepreneurs*.

## The Banking Sector

In most of the paper, we will assume a market structure with many (an infinite number of) infinitely lived banks that compete with each other to sign contracts with the entrepreneurs. *Banks can commit to honor long term contracts, and competition occurs only at the time of signing up entrepreneurs.* Each bank has a sufficiently large array of customers to fully diversify the idiosyncratic shocks. A bank's sole objective is to maximize the expected present value of each and every contract. For the purposes of this work, differences across banks or frictions in their liabilities and assets portfolios are of no relevance and shall be omitted altogether.

In the beginning of every period all the banks compete to sign up members of the new born cohort. Each of the potential entrepreneurs draws his initial productivity  $z_0$  from the distribution  $F_z(\cdot)$ . The realization  $z_0$  is also observed by banks before bidding for each entrepreneur. This

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<sup>11</sup>The use of the term underground is consistent with our assumption that once an agent is using that technology he cannot be enforced to pay any liability.

is important because  $z_o$  will determine whether the agent becomes an entrepreneur or not and the contract received.

## The Enforceability Problem and the Liquidation Option

Given the previous assumptions, an active entrepreneur always has the option of switching to the underground. The key assumption that separates this economy from a standard frictionless environment is that there is no enforcement device that directly rules out this option from entrepreneurs. On the contrary, I will assume that banks can fully commit to honor their contractual obligations. Therefore, the analysis centers on the incentive problems of entrepreneurs.

There are two possible ways in which entrepreneurs can walk away from the productive technology. First, they can opt out in the beginning of the period. In this alternative they can access the underground technology immediately, attaining an utility of  $U$ . On the other hand, they can leave in the middle of the period. In this case he will miss the underground technology for that period, but he can seize all the working capital  $k$  under his control. Therefore, the default option depends on  $k$ :

$$V_d(k) \equiv k + \beta(1 - \delta)U$$

Given his inability to commit, these outside options to the entrepreneur will effectively restrict the contracts between banks and entrepreneurs. The first option imposes a minimum on the utility of active entrepreneurs.

Finally, by scrapping the plant they can recover an amount  $L$  of the set up cost, where,  $L \leq K_o$ . Besides the present value of  $\{k^i\}$ ,  $L+U$  determines the opportunity cost of maintaining a productive technology in operation. However, as opposed to  $L$ ,  $U$  will also have a separate effect as it constrains the set of incentive compatible plans between entrepreneurs and banks.<sup>12</sup>

## Bond Prices

The opportunity cost of the resources used by banks is determined by the interest rate they face. I assume that the price of one-period real bond  $q_t$ , for all  $t$ , follows an exogenous Markov process  $\{q\}$ . The following is assumed:

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<sup>12</sup>Alternatively, one could assume that in addition to the working capital, the entrepreneur can also seize a fraction of the value of the plant,  $\theta L$ . Here,  $V_d(k) = k + \theta L + \beta(1 - \delta)U$ . This extension does not complicate the analysis at all, but the credit constraints will be tighter in this world. The same remark applies if the agent can access the backyard technology in the same period of default, in which case  $V_d(k) = U + k$ . The results of these alternative cases can be replicated in the current set-up by manipulating the values of  $U$  and  $L$ .

**A. 3 (Q).**  $\{q\}$  is a Markov stationary, ergodic process, with finite support  $Q = \{q_1, q_2, \dots, q_m\} \subset (0, 1)$ ,  $m < \infty$ . The transitions are denoted  $P_q(\cdot, q)$  and the unique invariant c.d.f. by  $F_q(\cdot)$ . Moreover,  $\{q\}$  has positive persistence in the same sense used in **P**.

In this environment, individual entrepreneurs are affected by a common aggregate shock  $q$ . Two alternative interpretations can be given. The first is that this is a small economy open to international capital markets. The second takes this as an economy in which the government follows a rule for the interest rate such that the interest rate is a stochastic interest rate policy rule. An example of the second interpretation is presented in the appendix.

The following assumption, whose payoff will be evident later, is sustained in all paper:

**A. 4 (I).** *Entrepreneurs are more impatient than banks:*  $\beta \leq \min\{Q\}$

Taking the interest rates an exogenous stochastic process has the obvious disadvantage of omitting the feedback effects of its shocks. But, in addition of gains in analytical and computational tractability, the strategy here has the pedagogical value of isolating the effects of shocks in  $q$  on the macroeconomic structure of the industry.

## 4 Economies with Long Term Contracts

In this section we assume that the competition of banks is only in terms of signing up the new potential entrepreneurs. To save on notation, the exposition specializes to relationships initiated at period  $t = 0$ . A dynamic contract specifies, for every period the amount of capital advanced from the bank,  $k_t$ , and the repayment,  $r_t$  from the firm as functions of the history of the relationship. In each period, the idiosyncratic productivity  $z_t$  and the economy-wide discount factor  $q_t$  are observed prior to setting  $(k_t, r_t)$ . The allocations of each period will be functions of  $h^t \equiv \{z_s, q_s : 0 \leq s \leq t\}$ , the history of shocks in the relation. Denote  $H^t$  the set of all feasible histories  $h^t$ ,  $\forall t \geq 0$ , and  $\mathcal{H}^t$  the product space form by  $\mathcal{H}$ , the power set of  $Z \times Q$ .

A dynamic contract is a sequence of  $\mathcal{H}^t$ -measurable functions  $\sigma \equiv \{\sigma^t : t \geq 0\}$ ,  $\sigma^t : H^t \rightarrow \mathbb{R}_+ \times \mathbb{R}_+$ , of the form  $\sigma_1^t(h^t) = k_t$  and  $\sigma_2^t(h^t) = r_t$ , indicating how much working capital is advanced and the amount re-paid by the firm. Contracts must satisfy several constraints. First, the non-negativity of entrepreneur's consumption imply a form of *limited liability constraint*:

$$\sigma_2^t(h^t) \leq z_t f(\sigma_1^t(h^t)), \quad \forall h^t \in H^t, \forall t \in \mathbb{N} \quad (1)$$

The possibility of default also constraints the admissible set of  $\sigma$ . The transitions  $P_q, P_z$  and the functions  $\sigma$  will define the payoffs for each party of the relationship. For each  $h^t \in H^t, t \in \mathbb{N}$ ,

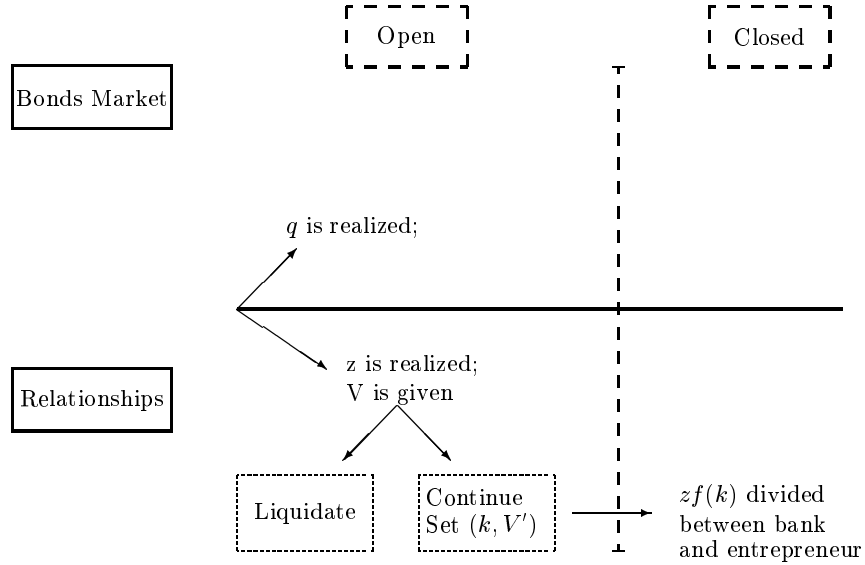
$P_q, P_z$  define the sequence of probability kernels,  $\mu^s(\cdot, h^t)$  on the measurable spaces  $(H^s, \mathcal{H}^s)$  for  $s \geq t$ . These kernels define the evolution of the history of each relationship. Indeed, at history  $h^t$  the continuation value for the entrepreneur implied by  $\sigma$  is given by

$$V_\sigma^t(h^t) \equiv \sum_{s \geq t} (\beta(1 - \delta))^{s-t} \int_{H^s} [z_s f(\sigma_1(h^s)) - \sigma_2(h^s)] \mu^s(dh^s, h^t) \quad (2)$$

Thus, the inability of entrepreneurs to commit imposes the constraints

$$V_\sigma^t(h^t) \geq U \text{ and } , \quad V_\sigma^t(h^t) \geq V_d(\sigma_1(h^s)) \quad \forall h^t \in H^t, \quad \forall t \in \mathbb{N} \quad (3)$$

The timing between the investment of working capital, collection of repayments, and the bond markets is needed to define the net payoff for banks. The following picture displays the timing used in this model.



#### Within-Period Flow of Events

The investment of working capital occurs at the beginning of the period, when the bonds market of the period  $t$  is open, but the bank collects the repayments only in the beginning of  $t + 1$ . The net present value of the payout for the bank in given history  $h^t$  is  $q_t \sigma_2^t(h^t) - \sigma_1^t(h^t)$ . Note also that because they access bonds markets, the discount factor for banks is  $q_t$ . Therefore, the value of a continuing relationship for the bank is

$$R_\sigma^t(h^t) \equiv \sum_{s \geq t} (1 - \delta)^{s-t} \int_{H^s} \left( \prod_{j=t}^{s-1} q_j \right) [q_s \sigma_2^s(h^s) - \sigma_1^s(h^s)] \mu^s(dh^s, h^t), \quad \forall t \quad (4)$$

This equation assumes that if the entrepreneur dies, the bank cannot seize,  $L$ , the liquidation value of the plant. Banks can always liquidate the firm. Commitment does not force banks to maintain the plants operating in all the eventualities, but instead to honor the utility entitlement to the entrepreneur dictated by the contract in every node of the relationship. Thus, in the case of liquidating the firm, the value of the bank is

$$L - [V_\sigma^t(h^t) - U] \quad (5)$$

Notice that the bank needs only to compensate the difference between  $U$  and  $V_\sigma^s(h^s)$ , which is simply  $V_\sigma^s(h^s) - U$  in units of the good and because of the linear utility assumption.

Clearly, an active firm will continue operating if and only if

$$R_\sigma^t(h^t) \geq L - [V_\sigma^t(h^t) - U] \quad (6)$$

Otherwise the firm is scrapped.  $\Sigma$  will denote the set of all incentive compatible allocations with elements  $\sigma$ . Notice that, despite the potential non-convexities originated by the scrapping option, we have ignored randomizations in the description of the contract. We will explicitly consider the randomizations in the recursive formulation, and verify that all non-trivial randomizations are relevant only outside the equilibrium allocation. Thus, the description here is without loss of generality.

## Initialization of Contractual Relationships

An optimizing bank will design  $\sigma$  so as to maximize his profits, but by competition, at the time of activation, agents would not sign unless they receive the best, feasible, incentive compatible contract. Thus, in equilibrium, entrepreneurs will be entitled the best initial utility  $V_\sigma^0(h^0)$ . The initial level of utility depends on the idiosyncratic characteristics  $z_0$  of each entrepreneurs and the economy-wide discount factor  $q$ . For each initial history  $h^0 = (z, q) \in Z \times Q$  let

$$\Omega(h^0) \equiv \{y \geq U : \sup_{\sigma \in \Sigma} \{R_\sigma^0(h^0) - K_0 - y\} \geq 0\} \quad (7)$$

$\Omega(h^0)$  is the set of promised utilities to the entrepreneurs that permit non-negative payoffs to banks. This set can be empty, in which case the agent does not become active. Otherwise he is activated with an utility entitlement of

$$V_\sigma^0(h^0) = \sup\{\Omega(h^0)\} \quad (8)$$

As show below, the *sup* operator insures that the contract is established is immune to future renegotiations.

## 4.1 A Recursive Formulation

The Markov nature of the shocks  $\{z\}$  and  $\{q\}$  allows the use of recursive methods to solve for the contracting problem. Following the initial insight of Spear and Srinivastava, we can specify the problem in terms of constructing rules for updating the continuation values *for the entrepreneur*,  $V$ . The vector  $(V, z, q)$  is the *state of individual relationships*, in the sense that it summarizes the history of it and contains all the relevant information for the future evolution of the relationship. The state has two components that evolve exogenously but the stochastic process followed by  $V$  is derived endogenously by the contract design. Given  $(V, z, q)$ , the bank must decide the amount of working capital  $k$ , the amount repaid by the firm  $r$ , and the continuation values for each possible realization in the next period. This is to say, given current state  $(V, z, q)$  the contract specifies for the next period the values  $G_{z',q'}(V, z, q)$  for the continuation utility of the entrepreneur for each and every  $(z', q')$ .

### 4.1.1 Ongoing Relations

Let  $C(V, z, q)$  be the cost (in expected present value) of providing an active entrepreneur with a utility level  $V$  when the market discount factor is  $q$  and the productivity of his plant is  $z$ . It  $C$  is negative the bank obtains positive payoff from the relationship.  $C$  can be positive which implies that the bank has to put net positive resources in the relationship to deliver the value  $V$ . The objective of the bank is to minimize this function. First, in case of liquidating the plant the net cost is  $V - U - L$ . If the plant continues, the period cost for the bank is  $-qr + k$  and commits to face the cost for the next period. Hereafter we use the shorthand

$$E[V^1] = \sum_{z',q'} G_{z',q'}(V, z, q) P_z(z', z) P_q(q', q) \quad (9)$$

for expected future utility entitlements, and

$$P(z', q'|z, q) \equiv P_z(z'|z)P_q(q'|q) \quad (10)$$

for the conditional probabilities.

For all  $(V, z, q) \in [U, \infty) \times Z \times Q$ ,  $C$  consider the following Bellman Functional Equation:

$$C(V, z, q) = \min \left\{ V - U - L, \min_{G_{z', q', k, r}} \left\{ k - qr + q \sum_{z', q'} C(G_{z', q', k, r}, z', q') P(z', q'|z, q) \right\} \right\}$$

subject to the constraints

$$\begin{aligned} U &\leq V_{z', q'} && \text{(Participation)} \\ k + \beta(1 - \delta)U &\leq V && \text{(No Default)} \\ zf(k) - r + \beta(1 - \delta)E[V'] &\geq V && \text{(Promise Keeping)} \\ r &\leq zf(k) && \text{(Limited Liability)} \end{aligned}$$

It is convenient to eliminate from the problem all the intra-temporal decisions. Let  $\pi(z, q)$ ,  $k^u(z, q)$  denote respectively the maximum profits from the technology and the unrestricted optimal use of working capital when enforcement problems are not binding, i.e.

$$\pi(z, q) \equiv \max_{y \geq 0} \{ qzf(y) - y \} \quad k^u(z, q) \equiv \arg \max_{y \geq 0} \{ qzf(y) - y \} \quad (11)$$

Under  $\mathbf{F}$   $\pi$  and  $k^u$  are  $k^u(z, q) = (zq\alpha A)^{1/(1-\alpha)}$  and  $\pi(z, q) = \Theta(zq)^{1/(1-\alpha)}$ , respectively, where  $\Theta$  is a positive constant that depends on  $(A, \alpha)$ .

We are interested in economies where the no default constraint might be binding. In those cases, the firm will use less working capital than the optimal level. We will say that the firm is *credit constrained in the intensive margin*. More formally, for any  $V \in [U, +\infty)$  the maximum level of capital that is sustainable with no default is

$$k^r(V) \equiv V - \beta(1 - \delta)U \quad (12)$$

It can never be optimal to use  $k$  above the unrestricted optimum; for any  $(V, z, q)$  the amount effectively used by continuing firms equals

$$k(V, z, q) = \min \left\{ k^u(z, q), k^r(V) \right\} \quad (13)$$

The enforcement-constrained surplus  $S$  is

$$S(V, z, q) = \begin{cases} \pi(z, q) & \text{if } V \geq V^u(z, q) \\ qzf(k(V)) - k(V) & \text{otherwise} \end{cases} \quad (14)$$

Several properties of  $S$  will be useful in characterizing the optimal allocation  $\sigma$ :

**Proposition 1.** *Under the assumption **(F)**, the function  $S$  is strictly increasing, and supermodular in  $(q, z)$ . For  $V \leq V_d(k^*(z, q))$ ,  $S(V, z, q)$  is strictly increasing, strictly concave, and strictly supermodular in  $(V, z, q)$ .*

<sup>13</sup> *Moreover,  $S(V, z, q)$  is globally concave and continuously differentiable in  $V$ .*

Using  $S$  we can eliminate  $k, r$ , and focus on the choice of the stochastic difference equation  $V_{z'q'} = G_{z'q'}(V, z, q)$  followed by the endogenous state and given by the optimal policy function in the recursive problem. First, the Promise Keeping constraint with equality implies that  $r = zf(k) + \beta(1 - \delta)E[V'] - V$ . We verify below that one can take the promise keeping constraint to hold with equality because in equilibrium, the relevant allocations are re-negotiation proof ( $C$  is strictly increasing in  $V$ ). Plugging this expression into the Non-Negativity of consumption, it becomes

$$\beta(1 - \delta) \sum_{z', q'} V_{z'q'}(V, z, q) P(z', q' | z, q) \leq V$$

The limited liability of the firm impose a ceiling on how fast  $\{V_t\}$  can grow. For each  $V \in [U, +\infty)$ , define the feasible set of continuation values  $\Gamma(V, z, q)$  by

$$\Gamma(V, z, q) = \left\{ y : Z \times Q \rightarrow [U, +\infty) \text{ s.t. } \sum_{z', q'} y(z', q') P(z', q' | z, q) \leq \frac{V}{\beta(1 - \delta)} \right\} \quad (15)$$

a convex and compact set. The net payoff for the bank becomes

$$k - qr = k - q[zf(k) + \beta(1 - \delta)E[V'] - V] \quad (16)$$

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<sup>13</sup>In the case twice continuously differentiability, a function  $f(x, y)$  is supermodular (submodular) in  $(x, y)$  if  $\frac{\partial^2 f}{\partial x \partial y} \geq (\leq) 0$ . The notions of submodularity and supermodularity are not restricted to differentiable functions a fact that makes them particularly useful in the context of dynamic where the differentiability results are rather limited. (Super)submodularity notions can be used for conditions of complementarity. Indeed, if  $f$  is submodular in  $(x, y)$ , then an increment in  $y$  reduces the marginal cost of  $x$ . See the appendix for a brief discussion of supermodular and submodular functions and Topkis [51] for a complete treatment.

which takes the form of  $-S(z, q, V) + q[V - \beta(1 - \delta)E[V']]$  once the optimal incentive compatible choice of  $k$  is plugged in. These substitutions transform the Bellman Equation to,

$$C(V, z, q) = \min \left\{ V - U - L, \quad Cc(V, z, q) \right\} \quad (17)$$

where

$$Cc(V, z, q) = \min_{y \in \Gamma(V, z, q)} \left\{ -S(z, q, V) + q[V - \beta(1 - \delta)E[y]] \right. \\ \left. + q(1 - \delta) \sum_{z', q'} C(y_{z'q'}(V, z, q), z', q') P(z', q' | z, q) \right\} \quad (18)$$

Finally, note that the option of liquidation introduces a non-convexity in the problem for ongoing relationship. Such non-convexity indicates gains from trade between the bank and the entrepreneur that are not being exploited. Randomizing the exit decisions eliminates those inefficiencies and the problem becomes convex. In turn, convexity greatly pays off in terms of sharp characteristics of the allocations and firm dynamics.

A risk neutral entrepreneur with utility entitlement  $V$  accepts any lottery that offers  $V^0$  with probability  $\lambda$  and  $V^1$  with probability  $1 - \lambda$  as long as  $\lambda V^0 + (1 - \lambda)V^1 \geq V$ . Given  $V$  the admissible set of those lotteries is

$$\Gamma_E(V) = \left\{ \lambda \in [0, 1], V^1 \geq V^0 \geq U : \lambda V^0 + (1 - \lambda)V^1 \geq V \right\} \quad (19)$$

where, the normalization  $V^0 \geq V^1$ , is obviously with no loss of generality. We can take that the entrepreneur is liquidated if he loses the lottery, and this is without loss of generality because banks and entrepreneurs will only use optimal randomizations. Therefore when the entrepreneur loses he is liquidated at utility value  $V^0$  and if he wins he remains active with an entitlement  $V^1 \geq V$ .

The Bellman Equation becomes that describe the optimal decisions of active firms is:

$$C(V, z, q) = \min_{(\lambda, V^0, V^1) \in \Gamma_E(V)} \left\{ \lambda(V^0 - U - L) + (1 - \lambda)Cc(V^1, z, q) \right\} \quad (20)$$

where  $Cc$ , the cost of continuing the relationship is as given above by equation 18.

Given the value function  $C$ , the initialization of relationships is also simplified. Let  $\Omega(z, q) = \{y \geq U : C(y, z, q) + K_0 \leq 0\}$ . If  $\Omega(z, q)$  is not empty the entrepreneur will become active and

with an initial entitlement of  $V_0(z, q) = \sup\{\Omega(z, q)\}$ . Otherwise, the agent will never be an entrepreneur. The *sup* operator is needed because the function  $C$  may not be monotone. Taking the *sup* insures the renegotiation proofness of the allocations (see below).

## 4.2 Optimal Contracts and Firm Dynamics

This subsection characterizes the value function  $C$  and the optimal policy correspondence  $G$ . The detail of the proofs and the notation required for them are relegated to the appendix. First, we can verify that the problem is well defined as there is a unique  $C$  that solves the functional Bellman Equation.

**Proposition 2.** *Let  $\Gamma_E, \Gamma, S$ , be as defined above. If conditions **F,P,Q** hold, the B.E. has a unique solution  $C : R_U \times Z \times Q \rightarrow R$ . The function  $C$  is globally decreasing in  $z$  and it is strictly decreasing in the regions where the probability of liquidation is less than one. Moreover,  $C$  is globally convex in its first argument.*

*Proof.* See appendix. □

That  $C$  is decreasing in  $z$  is very intuitive: the cost of providing any level of utility to the entrepreneur is a decreasing function of the productivity the technology under his control, as more resources can be expected from it in the current and subsequent periods. If the optimal decision were to liquidate, the cost is invariant w.r.t.  $z$ . Exit randomizations suffice to make a  $C$  convex function in its first argument. Because  $S$  is concave,  $Cc$  is convex whenever the function  $C$  under the expectation is convex; taking the optimal randomization, the LHS of the equation is convex whenever the function  $Cc$  in the RHS is convex.

Convexity is highly desirable analytically. First of all, it implies that  $C$  is everywhere subdifferentiable, and almost everywhere differentiable in the first argument. We use the notation  $\partial C$  denote the *subdifferential* of  $C$ , where  $\partial C(V, z, q)$  indicates the set of subgradients of  $C$  w.r.t. the first argument at the point  $(V, z, q)$ .  $C$  is differentiable w.r.t.  $V$  at  $(V, z, q)$  iff  $\partial C(V, z, q)$  is a singleton.

As we will see below, for a relevant region, the optimal policy will be at the corners. Then the usual manipulations a la Benveniste-Scheinkman, for the differentiability of  $C$  are of limited use. Yet, characteristics of  $\partial C$  will help characterizing the optimal policies. Indeed, we can easily verify that:

**Proposition 3.** *Given the conditions in the previous proposition, the subdifferentials  $\partial C, \partial Cc$  are such that  $\partial C(V, z, q) \leq 1$  and  $\partial Cc(V, z, q) \leq q$  for all  $(V, z, q) \in R_U \times Z \times Q$ . Moreover, if for given  $(z, q)$ ,  $\exists V^* \in R_U$  with  $C(V^*, z, q) < -L$ ,  $\partial C(V, z, q)$  is also bounded above by  $q$*

*Proof.* See appendix. □

Therefore the *joint surplus*  $V - C(V, z, q)$  is increasing in  $V$ , and strictly increasing in neighborhoods where  $C_c = C$ . This is not surprising as a higher  $V$  helps coping with the default problems of entrepreneurs, the only incentive issue in this economy. However, the fixed point  $C$  is not necessarily increasing in  $V$ .<sup>14</sup> This is troublesome because if  $C$  is decreasing in  $V$ , it is not valid to take the promise keeping constraint with equality. Indeed, *both* parties would be willing renegotiate up the value of  $V$  and be better off. We can rule out this possibility, but only after we characterize the initialization of the relationship. This is, while the value function  $C$  can have segments where it is decreasing  $V$ , in equilibrium, those segments are never reached.

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The submodularity of  $S$  in  $(V, z)$  indicates a form of complementarity of the productivity of the technology with the value for the entrepreneur. The larger is the value of the relationship for the entrepreneur, the lesser are the temptations for him to default. Then, higher productivities could be met with larger employment of working capital. The supermodularity of  $S$  suffices for the submodularity of  $C$  when  $\{z\}$  has positive persistence, which holds by assumption **P**.

Therefore,

**Proposition 4.** *Assume that the conditions **F,Q,P** hold. Then, the fixed point  $C$  is a submodular function in  $(V, z)$ .*

*Proof.* See the appendix. □

The convexity and submodularity of the value function can be used to characterize the optimal allocation  $\sigma$  and the implied firm dynamics. First we analyze the continuation decisions (the evolution of  $V$  given that the firm survives that period). Then we characterize the entry and exit decisions.

We also verify below that the non-trivial randomization are outside the equilibrium allocation and that the equilibrium allocations are re-negotiation proof.

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<sup>14</sup>To see this, observe that the period return  $-S(V, z, q) + q[V - \sum_{z', q'} G_{z', q'}(V, z, q)P(z', q'|z, q)]$  for the bank it is not necessarily increasing in  $V$ : On the one hand, for any given policy  $V_{z', q'}$ , the higher the resources consumed by the entrepreneur the higher is the entitlement  $V$ ; on the other hand, however, lower values of  $V$  will constraint more the surplus  $S$  from the technology. If the second dominates the period return of the bank can be increasing in  $V$ . Moreover, even if restriction on  $U, f, Z, Q$  can assure that  $-S(V, z, q) + qV$  be increasing in  $V$ , the fixed point  $C$  may have decreasing regions w.r.t.  $V$  because the feasible set is increasing in  $V$  in the set inclusion sense (i.e.  $V_0 < V_1 \Rightarrow \Gamma(V_0, z, q) \subset \Gamma(V_1, z, q)$ ).

<sup>15</sup>For the case of  $Q = \beta$  Albuquerque and Hopenhayn [1] prove that the *joint surplus*  $V - C(V, z)$  is increasing in  $V$ . This does not eliminate the renegotiation problem as it does not imply that the payoff for the bank –the part with full commitment and designing the contract– is decreasing in  $V$ .

## Continuation policies

Let  $G_{z',q'}(V, z, q)$  denote the entrepreneur's utility entitlement profile for the next period, i.e. for each realization  $z', q'$  in that period given a current state  $(V, z, q)$ . The first order conditions – sufficient and necessary for a convex problem – for the optimal  $G_{z',q'}(V, z, q)$  are

$$\beta(1 - \mu_2(V, z, q)) + \mu_1(z', q'; V, z, q) \in \partial C(G_{z',q'}(V, z, q), z', q'), \quad \text{for all } z', q' \quad (21)$$

$$\mu_1(z', q'; V, z, q) \geq 0; \quad G_{z',q'}(V, z, q) \geq U; \quad (\text{and at least one with } =) \quad \text{for all } z', q' \quad (22)$$

$$\mu_2(V, z, q) \geq 0, \quad \beta(1 - \delta) \sum_{z', q'} G_{z',q'}(V, z, q) P(z', q' | z, q) \leq V, \quad (\text{and at least one with } =) \quad (23)$$

where  $\mu_1(z', q'; V, z, q)$  and  $\mu_2(V, z, q)$  are  $(\#Z \times \#Q) + 1$  (scaled) Kuhn-Tucker multipliers. The first set of multipliers makes sure that the bank does not assign a value below the underground's value for the entrepreneur while the last multiplier ensures that the his consumption is non negative. Manipulation of these FOCs provide restrictions on the valid  $G_{z',q'}(V, z, q)$ . First we can verify that  $G_{z',q'}(V, z, q)$  is non-decreasing in  $V$ :

**Proposition 5.** *Let  $\sqsubseteq$  denote the higher set partial ordering.<sup>16</sup> The policies  $G_{z',q'}(V, z, q)$  are non decreasing in  $V$ , in the sense that for any  $(z, q)$  and  $V_0 < V_1$ ,  $G_{z',q'}(V_0, z, q) \sqsubseteq G_{z',q'}(V_1, z, q)$  for all  $(z', q')$*

*Proof.* See appendix. □

This result is hardly surprising as  $V$  imposes only an upper bound on  $E[V']$ . Therefore, if it does not bind for a  $V_0$ , it does not bind for  $V \geq V_0$ . In periods when  $V$  binds, the entrepreneur has zero consumption and the expected utility entitlement for the next period is as high as possible. This is very intuitive, given the one-sided nature of incentive problem studied here. Initially the value of the relationship for the entrepreneur must grow as fast as possible because it enhances the future profits extractable from the technology. The result obtains regardless of how  $Q$  compares with  $\beta$ , in sharp contrast with the results obtained by O.Hart and J.Moore [35] in models with two-sided imperfect commitment.<sup>17</sup>

<sup>16</sup>  $A \sqsubseteq B$  if for  $x \in A$ ,  $y \in B$   $x \vee y \in B$  and  $x \wedge y \in A$ .

<sup>17</sup> In Hart-Moore's model the bank cannot commit not to renegotiate the contract with the entrepreneur. In their model, if  $\max\{Q\} < \beta$ , the optimal contract is the slowest repayment schedule, and if  $\beta < \min\{Q\}$ , it is the fastest repayment schedule. For  $\beta = Q$  there is an indeterminacy as a continuum of contracts are optimal.

The optimal contract not only must specify the evolution of the entrepreneur's utility entitlement over time but also across realizations of the idiosyncratic and aggregate shocks. The submodularity  $C$  provides another clear restriction of the optimal continuation profile:

**Proposition 6.** *Let  $(V, z, q) \in R_U \times Z \times Q$ .  $C$  is submodular, then  $G_{z',q'}(V, z, q)$  is increasing in  $z'$ . Moreover, if  $G_{z',q'}(V, z, q) > U$ , then it is strictly increasing.*

*Proof.* See appendix. □

The economics of this result is very simple. A higher productivity increases the optimal use of working capital. Yet, a higher advance of credit increases the temptations of the entrepreneur to default. Foreseeing this, it is optimal to increase the utility entitlement in a positive relation with the productivity of the states. Moreover, a higher productivity signals higher productivities thereafter, and a higher value  $V'$  will relax the constraints for setting  $V''$ ,  $V'''$ , ..... and so on. Symmetrically, low realizations of  $z$  reduce the value of the relationship for the entrepreneur, but this is a result of its optimal design. Indeed, we will see below that when the entrepreneur is liquidated, the bank does not have to compensate the entrepreneur and simply seizes the plant. Before turning to exit decisions, it is convenient to characterize further the continuation policies. Specifically, we will provide conditions under which they are bounded.

By inspection of the first order conditions and of the form of  $Cc$  is not hard to see that if  $\exists q^0 < \beta$ , then the bank would find it optimal to prescribe that, regardless of the current state  $(V, z, q)$ , either  $G_{z',q^0}(V, z, q) = U$  (liquidation) or  $G_{z',q^0}(V, z, q)$  to be as large as possible. This is to say, in those states in which the bank is more impatient than the entrepreneur, the bank would postpone the delivery of dividends to the entrepreneurs *ad infinitum*. We rule out this degeneracy by imposing condition **I**.

The participation constraint of the entrepreneur provides a lower bound  $G \geq U$ . Under assumption **I** we proceed to verify that  $G$  has a finite upper bound. The boundedness of  $G$  brings about obvious computational advantages.

First, assume that for a given state  $(V, z, q)$  the choice of  $G_{z',q'}(V, z, q)$  is interior in the sense that  $V$  is not binding ( $\mu_2(V, z, q) = 0$ ). In these cases, one can use Benveniste-Scheinkman and obtain

$$\frac{\partial C(V, z, q)}{\partial V} = -\frac{\partial S(V, z, q)}{\partial V} + q$$

Now, let us assume that the choice of  $G_{z',q'}(V, z, q)$  is such that  $\mu_2(G_{z',q'}(V, z, q), z', q') = 0$ . Using the functional form in **F** we obtain a unique value  $M^*(z', q')$  that solves the FOC, given by

$$M^*(z, q) = \beta(1 - \delta)U + \left[ \frac{\alpha z}{1 + \frac{1-\beta}{q}} \right]^{\frac{1}{1-\alpha}} \quad (24)$$

One cannot conclude however that  $M^*(z', q')$  is an upper bound of  $G_{z', q'}(V, z, q)$  as we have not yet ruled out that it binds in the subsequent choice of  $G$ . This is to say, it might be the case that for some  $z^0, q^0$

$$M^*(z^0, q^0) < \beta(1 - \delta) \sum_{z', q'} M^*(z', q') P(z', q' | z^0, q^0)$$

Yet, the boundedness of  $M^*$  leads to an upper bound on  $G_{Z, Q}(\cdot)$ . One can find a finite function  $D : Z \times Q \rightarrow R_{++}$  such that the policies always lie within  $[U, D(z', q')]$ :

**Proposition 7.** *Assume that **I** holds. Then the continuation policies are bounded. Specifically,  $U \leq G_{z', q'}(V, z, q) \leq D(z', q') \leq M(\bar{z}, \bar{q}) < \infty$  for all  $z', q', V, z, q$ . The function  $D$  is the unique fixed point that solves*

$$D(z, q) = \max \left\{ M^*(z, q), \quad \beta(1 - \delta) \sum_{z', q'} D(z', q') P(z', q' | z, q) \right\}$$

Moreover,  $D$  is strictly increasing both arguments and  $D(\bar{z}, \bar{q}) = M^*(\bar{z}, \bar{q})$

*Proof.* See appendix. □

Therefore, for the computations one can restrict attention to the domain  $[0, M(\bar{z}, \bar{q})]$ . Furthermore, one can safely assume that the policy correspondence  $G$  is indeed a continuous function, which is guaranteed by the following proposition.

**Proposition 8.**  *$Cc(\cdot, z, q)$  is strictly convex and strictly submodular in  $(V, z)$  if either  $\mu_2(V, z, q) > 0$  or  $V < M(z, q)$ . Moreover, for any  $(V, z, q) \in R_U \times Z \times Q$ , the policy correspondence  $G_{z', q'}(V, z, q)$  is a continuous function of  $V$ .*

*Proof.* See the appendix. □

Therefore, the policy correspondence  $G_{Z, Q}(\cdot)$  is singled valued, and that indeed, it varies continuously on the endogenous state  $V$ . Therefore, the previous monotonicity results, not only can be rewritten with the partial ordering  $\leq$  instead of the set ordering  $\sqsubseteq$ , but can be taken to be strict as long as  $G > U$ .

## Liquidation Decisions

We now turn the attention to the *extensive margin* decision of whether to liquidate or continue a relationship. It turns out that, as long as  $V > U$ , there are only two possibilities: either the entrepreneur continues with probability one or there is a liquidation lottery. This is because either  $Cc$  is always below the value of liquidation or they cross at some finite value,  $V^{**}$ , a fact that can easily be established because the slope of  $Cc$  is uniformly bounded by  $q < 1$  while the slope for the liquidation is always 1. Therefore, the envelope of  $Cc$  and  $V - U - L$  is  $Cc$  in its all region or forms from a convex combination.

This suggests that an entrepreneur is never liquidated without a lottery. But we will soon see that exactly the opposite is true. From risk neutrality, it follows that entrepreneurs losing the lottery must obtain the lowest utility  $U$  and exit. If not,  $C$  would have a segment with slope 1 and then a segment with slope  $q$  violating convexity.

When  $V = U$ , the bank recommends exit, and seizes the plant, obtaining the liquidation value  $L$ . Such process resembles bankruptcy in many respects, as the entrepreneur enters the pool in the underground with no assets left from the firm. More formally,

**Proposition 9.** *Fix  $(V, z, q)$ . If the optimal liquidation probability  $\lambda(V, z, q) \in (0, 1)$ , then,  $V^0 = U$  and there is a unique value  $V^1 < M^*(\bar{z}, \bar{q})$  that satisfies*

$$\frac{Cc(V^1, z, q) + L}{V^1} \in \partial Cc(V^1, z, q)$$

*The optimal liquidation probability,  $\lambda$ , is given by  $\lambda = \frac{V^1 - V}{V^1 - U}$*

*Proof.* See the appendix. □

Entrepreneurs winning the lotteries obtain a new utility entitlement  $V^1$  in a region where  $Cc$  is strictly convex. If that were not the case,  $Cc$  would have a constant slope of  $q$ . Generically, this implies that additional gains can be obtained by setting  $V^1$  even larger, and consequently,  $\lambda$  closer and closer to unity. We haven't found conditions to rule out degenerate lotteries  $V^1 = \infty$  and  $\lambda = 1$ . They are not present in the computations presented below.

Notice that for each  $(z, q)$  either  $\lambda(\cdot, z, q) = 0$  in all the domain  $[U, \infty)$  or there is a region  $V \in [U, V^1]$  such that  $\lambda(\cdot, z, q) > 0$ , and for  $V \in (V^1, \infty)$   $\lambda(\cdot, z, q) = 0$ . Indeed,

**Proposition 10.** *The liquidation probability function  $\lambda : [U, \infty) \times Z \times Q \rightarrow [0, 1]$  is decreasing in its first argument and is strictly decreasing whenever  $\lambda \in (0, 1)$ .*

The economics is also simple. The presence of both a fixed cost  $U + L$  of operating a plant and the complementarity of  $z, V$  implies that it is worth to operate the plant as long as the

level of operations is large enough. If the current value of  $V$  is above  $V^1$ , the scale of operations is large enough, and no randomization is needed; the plant continues operation with certainty. The lower the value of current  $V$  the larger is the risk of liquidation faced by the firm in a fair lottery with a constant winning and losing prizes of  $V^1$  and  $U$  respectively. Therefore, the model implies that the lower the value of  $V$ , the lower is the survival probability, and that conditional on surviving, the larger is the growth. Conditional on  $z, q$ , the lower  $V$  the larger is the variance of growth, unconditional on survival.

It seems that to characterize exit we need to examine the function  $\lambda(V, z, q)$ . This is unnecessary. By inspection of the first order conditions we can verify the following:

**Proposition 11.** *Non-Trivial Randomizations are outside the equilibrium allocation Fix any  $(V, z, q) \in \mathcal{R}_U \times Z \times Q$ , and  $V^1(\cdot)$  as defined above. Then, the optimal continuation mapping  $G_{Z \times Q}(\cdot)$  can be selected so that  $G_{z', q'}(V, z, q) \in \{U\} \cup (V^1(z', q'), D(z', q'))$  all  $z', q'$*

*Proof.* See the discussion in the text, and the details in the appendix. □

Therefore, no randomizations are required. To see this, notice that in the regions where randomizations occur, the slope of the cost function is constant. The first order conditions equate the marginal reduction in the current cost with the marginal cost of assigning the utility entitlement in every realization of the next period. If the limited liability constraint does not bind, the first term is just the discount factor  $\beta$  while the second is the would be a constant in the region of randomizations. If the discount factor is strictly higher, then the optimal plan requires  $G_{z', q'} = U$  while if it is strictly lower, then  $G_{z', q'} = V^1(z', q')$ . In case of equality, either extreme can be chosen, as the entrepreneur and the bank are indifferent in the timing of the transfers. In those knife-edge cases deterministic rules suffice.

Now, if the limited liability constraint binds, then there is a state  $z', q'$  in which the slope of the cost is strictly below the discount factor of the entrepreneur. Then for the cases where  $\beta \in \partial C(V^1(z'', q''), z'', q'')$ , the optimal solution is  $G_{z'', q''} = U$ . Even if the limited liability binds, there is a possibility that  $\beta(1 - \mu_2(V, z, q)) \in \partial C(V^1(z'', q''), z'', q'')$ . But again, the deterministic rule that assigns  $G_{z'', q''} = U$  achieves the optimum.

Clearly, a more binding limited liability prescribes a larger set in which the firm is liquidated in the next period.

## Initialization of Relationships

Competition prompts banks to offer the best feasible incentive compatible contract to entrepreneurs. Because banks need to finance the fixed cost  $K_0 > L$ , the *ex-ante* breaking-even

condition implies that the bank *ex-post* makes a strictly positive profit from the entering ventures. The function  $V_0(z_0, q_0)$  indicates the value of the entrepreneur at the time of signing the contract with the bank.

**Proposition 12.** *Let  $V_0 : Z \times Q$  as defined above. Given the assumptions **F,P,Q**, then  $V_0$  is strictly increasing in  $z$ . Moreover, given  $L$  if  $K_0 > L$  is large enough, for  $\beta$  close enough to 1 and  $\delta$  close enough to 0 then  $V_0$  is also strictly increasing in  $q$ .*

Entrepreneurs with a higher initial productivity draw  $z_0$  receive a higher value  $V_0(z_0, q_0)$  as the present value of the stream of profits is larger, and also because a larger value  $V_0$  will enhance the incentive-compatible allocation of credit. The second part requires that  $\beta$  and  $\delta$  close to 1 and 0, respectively only insofar they limit  $E_t[V_{t+1}]$  and then, in expectation,  $C$  is also negative in the next period. In such a case, higher interest rate unambiguously increases the cost and reduces the breaking-even value  $V_0(z_0, q_0)$ . With higher interest rates not only entering entrepreneurs receive a lower  $V_0$ , but also some entrepreneurs may no longer become activate.

In periods of high interest rates, the mass of firms entering is smaller. In those periods, the requirements of productivity  $z$  to activate are higher. As a result, the entering firms in periods of high interest rates are more likely to survive. On the other hand, the model has the observable implication that conditional on  $z$  and given the stationarity of  $F_z$ , the pool of entrants, entering firms in periods of high interest rates are smaller than firms that enter in periods of lower interest rates. The lower value  $V_0$  reduces the survival probabilities of those firms.

We close this section confronting our claim that in equilibrium the allocations are not subject to renegotiation. While, at the time of signing the contract,  $V_0(z, q)$  cannot lie in regions where  $C$  is decreasing in its first arguments, one might question whether the allocations in the future would entail possible Pareto improvements. If that is the case, the contracting parties will renegotiate a new contract and the studied allocations would no longer be relevant.

Competition implies that the initial entitlement  $V_0(z_0, q_0)$  must be located in a neighborhood where  $C(\cdot, z_0, q_0)$  is strictly increasing. Otherwise, the entrepreneur is not receiving the best feasible contract. But it is easy to verify that this fact along with the optimization underlying the function  $C$ , implies that, unless the utility entitlements are equal to the underground value  $U$ , they lay in regions where  $C$  is also strictly increasing in the first argument. Formally,

**Proposition 13.** *Fix  $(z, q)$  and assume that  $V \geq V_0(z, q)$ . Then, for every  $z', q' \in Z \times Q$ , either  $G_{z', q'}(V, z, q) = U$ , or  $\partial C(G_{z', q'}(V, z, q), z', q') > 0$*

*Proof.* See appendix for details. □

The economics captured by this proposition is very clear. The optimal contract foresees correctly the future. For the initial expected payoff of the bank to be strictly decreasing in the initial expected payoff of the entrepreneur, then it must be the case that no renegotiation can be feasible in any future date or contingency with positive probability. Otherwise, the contract is not optimally designed, as it can be strictly dominated by modifying it with the Pareto improving amendments. Therefore, once the contract is established, the rules specified in at the time of signing the contract.

In a more technical note, this result validates our using the promise keeping constrained with equality.

## 5 Decentralization With Short-Term Securities

In this section we show that the allocations attained by long term contracts can be replicated by sequences of one-period contingent contracts. Our argument follows closely Alvarez and Jermann [2] and to a lesser Kehoe and Levine [39]. As Alvarez and Jermann [2], we will examine environments where entrepreneurs trade in standard securities markets, but their portfolio choices are restricted, in our case, by solvency as well as borrowing constraints.<sup>18</sup> We show how to find the price system and the profile of constraints so that the allocation attained in these less sophisticated trading environments replicates.

In our environment *borrowing constraints* are needed in addition to solvency constraints. This extra requirement is the consequence of the added possibility of entrepreneurs to appropriate the working capital. Extra investments in notation are required by the liquidation option. The steps of this exercise are the following. First, we define the problem of a maximizing entrepreneur facing given arbitrary asset prices and solvency and borrowing constraints. Second, from the long term contract we construct a candidate profile for the constraints and the securities prices. In the third, ignoring the participation and default constraints, we verify that the agent find it optimal to choose the allocations from the long term contract, given the candidate prices and constraints. Finally, we verify that the allocation also solves the problem of the agent, even when facing the participation, exit, entry and default decisions.<sup>19</sup>

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<sup>18</sup>Kehoe and Levine [39] follow Prescott and Townsend by imposing the incentive constraints in the consumption sets.

<sup>19</sup>Albuquerque and Hopenhayn [1], following Bulow and Rogoff, [7] consider the case when defaulting entrepreneurs can maintain savings in other financial intermediaries and write down contingent contracts. The key difference is that we assume a strong form of *concertated action* on behalf of financial intermediaries: that defaulting agents are out of the entire financial system. This is, we assume that while financial intermediaries compete with each other in the securities market, they can commit not to lend and also to seize the savings of

Another decentralization scheme admissible in this environment is to simply interpret the previous environment as one in which the entrepreneur acquires long term, contingent liabilities from an intermediary, but that he can move from one bank to another as long as he pays back the remaining balance in the original relation. Banks break-even in expectation in every period, as the entrepreneur transfer his debt to the new bank which pays back to the original bank. This last interpretation is an immediate by-product of the construction of the candidate budget constraints.

Anyway, the exercises in this section yield an explicit link between the conceptual variable  $V$  in a relationship with a much more pedestrian form of collateral, the wealth of the entrepreneur in a decentralized setting.

### The Deterministic Case

The essence of the argument can be captured by the case with no uncertainty where  $\delta = 0$ , and normalize units so that  $z_t = 1$  all  $t$ . The results for the general case are straightforward extensions once the appropriate notation is in place.

Consider the problem of an entrepreneur that can borrow or lend using one-period assets in impersonal markets. The only asset is a one-period bond with price  $q$ . For now, ignore the occupational choice and the option of default. Assume that for unspecified reasons, the entrepreneur must satisfy solvency and borrowing constraints  $\hat{\mathcal{A}}, \hat{\mathcal{B}} = \{\hat{A}_t, \hat{B}_t\}$ . The solvency constraints indicate that in every period  $t$ , the wealth  $a_t$  of the agent must always be at least  $\hat{A}_t$  and the borrowing constraint restricts the external borrowing at period  $t$  to be at most  $\hat{B}_t$ . The only asset needed in this case is a one period bond, which price we take as  $q$ .

The problem  $\mathcal{P}(\hat{\mathcal{A}}, \hat{\mathcal{B}})$  of the entrepreneur, given  $q$  and  $\mathcal{A} = \{\hat{A}_t, \hat{B}_t\}$ , the arbitrary solvency and borrowing constraints is:

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any entrepreneur who, in any previous period, has defaulted, to *any financial intermediary*. This form of cooperation among intermediaries could be derived from an infinite horizon game among intermediaries. We can also find support for the assumption from the working of actual civil (common) courts, credit bureaus, etc, in actual economies. Indeed, information sharing arrangements among creditors are pervasive world-wide, f.g. Padilla and Pagano [46], even in developing countries. Indeed, creditors share information not only on credit performance but also on asset ownership by debtors. See f.g. Monge et. al [45] for a country study.

$$\max_{c_t, a_t, k_t} \sum_{t \geq 0} \beta^t c_t \quad (\mathcal{P}(\hat{\mathcal{A}}, \hat{\mathcal{B}}))$$

such that for all  $t$

$$a_t \geq A_t \quad (\text{Solvency Const.})$$

$$0 \leq k_t \leq a_t - A_t + \hat{B}_t \quad (\text{Borrowing Constraints})$$

$$0 \leq qc_t \leq a_t + q[f(k_t)] - qa_{t+1} + k_t \quad (\text{Sequential Budget Constraint})$$

The specification of  $\hat{\mathcal{A}}, \hat{\mathcal{B}}$  determines whether the agent can become active, the feasible level of operation, his consumption, etc. For example, to become active, the entrepreneur at  $t = 0$  it is necessary that  $\hat{A}_0 \leq -K_0$  so we can buy the plant, and that  $\hat{B}_0 > 0$  to make it operate in that period. Yet, we are not interested in the allocations from arbitrary  $\hat{\mathcal{A}}, \hat{\mathcal{B}}$ . The question is whether there is a profile, say  $\mathcal{A}, \mathcal{B}$ , in which the allocations in this far less sophisticated environment replicate  $\sigma$ .

Using the long term allocation,  $\sigma$  we can read off the implied net asset positions of the entrepreneur in the relation, as  $A_t \equiv C_t^\sigma$ , i.e. the value assumed by  $C_t$  along the optimal long term contract. The initial value for the sequence is given by  $A_0 = -K_0$ , i.e. the entrepreneur starts with net debt to finance the plant. The wealth is updated according to the repayments and working capital advances  $r_t^\sigma, k_t^\sigma$ , following the recursion:

$$A_t = -qr_t^\sigma + k_t^\sigma + qA_{t+1} \quad (25)$$

It is easy to see the infinitely lived contract can be replicated by rolling over a short term liability. Say, in the beginning of the period an entrepreneur owes  $A_t < 0$ . Then, he can borrow the amount  $A_t + k_t$  from another bank, and ask the new bank to pay his debt  $A_t$  to the old bank, freeing him from any that liability. In that period the entrepreneur would use the working capital  $k_t$ , produce  $f(k_t)$  and make the repayment  $r_t$  to the new bank; those transactions imply that the next period will begin with the agent having a debt (or deposit) of  $A_{t+1}$  in the new bank. At that point he can decide to remain or to move on with a different bank, and so on. Banks would be breaking-even in every period. It requires that no bank will lend to entrepreneurs who have previously defaulted. To cope with the incentive to switch to the underground sector and to appropriate of the working capital require the lower limits in the debt rolled over (for

participation) and upper limits on the borrowing (default with appropriation of  $k$ ). The obvious candidate for the borrowing constraint is simply  $\mathcal{B} = \{k_t^\sigma\}$ .

The allocation implied by  $\sigma$  uniquely solves  $\mathcal{P}(\mathcal{A}, \mathcal{B}, q)$ . Suppose that the economy is such that it is optimal to activate the firm in the long term contracts setting. Since  $A_0 = C_0 = -K_0$ , here the entrepreneur can also become active in the first period of life and by construction he can operate the plant up to  $k_0 = B_0 = k_0^\sigma$ . This holds for any period  $t$ ; the allocation implied by  $\sigma$  is feasible because by construction it and respects the constraints  $\mathcal{A}, \mathcal{B}$ . We can verify that indeed the agent cannot do better (see the appendix) because otherwise the contract used to design  $\mathcal{A}, \mathcal{B}$  could be improved upon, which is a contradiction with the definition of  $C$  or the initial value  $V_0$ . Finally, assume that the entrepreneurs can have the option of leaving without honoring previously accumulated debt or alternatively, leaving the system and appropriating the working capital advanced in the period, exactly as above. Would their participation and default decision be different in this trading environment with  $\mathcal{A}, \mathcal{B}$  with respect to the long term contract environment? Clearly no. While the profile of credit constraints allows infinitely many paths with nodes in which the entrepreneur would run away as the entrepreneur will disregard those paths as suboptimal. This is, for any period  $t$ , including the activation date  $t = 0$ , once the agent is placed along the optimal default free allocation from the long term contracts, he will disregard those other alternatives as suboptimal.

## The General Case

The case with uncertainty follows naturally from the previous considerations, but requires extra notation. Here, the solvency and borrowing constraints need to be specified as functions of the partial histories of the relationship. The steps of the argument are the same: define the problem of agents facing exogenously given solvency and borrowing constraints; from the long term contract find the candidate prices for the securities traded and the solvency and borrowing constraints; ignoring the participation and default constraints, it would be obvious that the allocation  $\sigma$  is optimal, and then argue that incorporating the default and participation options does not affect the arguments.

First, we look at the interpretation of the contract as rolling over a short-term financial liability. Here the entrepreneur in the first period obtains a loan of  $K_0$  from one bank to start up the plant, plus  $k_0$  for the working capital of that period. The financial system records his balance of  $-C(z_0, q_0) - k_0$  with that bank. Uncertainty present, the optimal one period contract will be contingent on the realization at  $t = 1$ . Clearly, the bank and the entrepreneur can sign a contract in which the balance of the latter is the value  $C^\sigma(z^1, q^1)$  attained from the long recursive problem solved in the previous section. This is, a one period contract in which the entrepreneur

commits to repay  $r\sigma_0(z_0, q_0)$  and the next period is the random variable  $C^\sigma((z_1, z_0), (q_1, q_0))$ . When the uncertainty of the next period is realized, the financial system records the debt of the entrepreneur. At that point, he is free to contract with that bank with a similar contract, or move on to another bank. If he opts for the latter, then he needs to pay for the balance  $C^\sigma((z_1, z_0), (q_1, q_0))$  to the old bank. The new bank pays off the debt to the old one, and now the credit bureau records that the balance of the entrepreneur is with the new bank. And so on. It is possible that  $C^\sigma((z_1, z_0), (q_1, q_0)) = -L$ ; in this case, the value of the plant is exactly the amount owed to the bank. We know from the long term contracts that happens when  $G_{z,q}(\cdot) = U$ . Then, the plant is seized by the bank to recover the debt. Finally, if the agent dies, no one can recover any resources from the technology and the balance is recorded as zero. By construction, as before, in each period the bank breaks even in expectation.

Along the lines of the previous subsection, we ask whether we can replicate the allocation of the infinite horizon contracts in centralized asset markets with endogenously designed exogenous constraints. A very significant simplification is that in recording the histories of the relationships we can ignore (non-trivial) liquidation randomizations because they are relevant only outside the equilibrium allocation. In addition to  $z, q$  shocks, the relationships also face the risk of the death of the entrepreneur, in which case no resources can be recovered. Moreover, we need to keep track of whether the agent has been previously liquidated, which implies, by assumption, that he cannot operate the productive technology anymore.

Denote  $d_t$  the realization of the death risk;  $d_t = 0$  indicates that the agent dies at period  $t$  and  $d_t = 1$  that he survives. Naturally,  $d^t$  denotes the partial history of the death risk and  $D^t$  the set of all partial histories.

The assets used by the entrepreneur to trade are contingent on the realization of the  $(z^t, q^t, d^t)$  shocks. Take  $p^*$  as arbitrary the Arrow prices, and  $\mathcal{A}, \mathcal{B}$  arbitrary solvency and borrowing constraints. Then,  $\mathcal{P}(\mathcal{A}, \mathcal{B}, p^*)$ , the problem of the agent becomes:

$$\max_{\{c,a,k\}} \sum_{\{t \geq 0\}} \sum_{\{Z^t \times Q^t \times D^t\}} \beta^t c_t(d^t, z^t, q^t) \mu_t(d^t, q^t, z^t | q_0, z_0) \quad (\mathcal{P}(A, \mathcal{B}, p^*))$$

such that, for all  $(d^t, q^t, z^t) \in Z_t \times Q^t \times D^t$  all  $t$

$$\begin{aligned} a_t(d^t, z^t, q^t) &\geq A_t(d^t, z^t, q^t) && \text{(Solvency Const.)} \\ 0 \leq k_t(d^t, z^t, q^t) &\leq a_t(d^t, z^t, q^t) - A_t(d^t, z^t, q^t) + \hat{B}_t(d^t, z^t, q^t) && \text{(Borrowing Constraints)} \\ a_t(d^t, z^t, q^t) + q_t[f(k_t(d^t, z^t, q^t)) - c_t(d^t, z^t, q^t)] - k_t(d^t, z^t, q^t) &\geq \\ \sum_{\{z_{t+1}, q_{t+1}, d_{t+1}\}} [p^*(d_{t+1}, z_{t+1}, q_{t+1} | z_t, q_t) a_{t+1}(d_{t+1}, z_{t+1}, q_{t+1}, z^t, q^t)] &&& \text{(Seq.Budget Constraint)} \\ c_t(d^t, z^t, q^t) &\geq 0 && \text{(Limited Liability)} \end{aligned} \quad (26)$$

We have ignored the choices after being liquidated. Following our line of argument, they are considered at the end.

We now construct our candidate price system and solvency and borrowing constraints. The natural candidate for Arrow prices are:

$$p_t^*(d^{t+1}, z^{t+1}, q^{t+1} | d^t, z^t, q^t) = \begin{cases} 0 & \text{if } d_{t+1} = 0 \text{ or} \\ & \text{or if } d_s = 0 \text{ for some } s \leq t \\ G_{z_s, q_s}(z^{s-1}, q^{s-1}, V_{s-1}(z^{s-1}, q^{s-1})) = U & \\ (1 - \delta) q_t P(z_{t+1}, q_{t+1} | z_t, q_t) & \text{otherwise} \end{cases}$$

The implied Arrow-Debreu prices are defined by the recursion

$$P_t^*(d^{t+1}, q^{t+1}, z^{t+1} | z_0, q_0) = p_t^*(d_{t+1}, z_{t+1}, q_{t+1} | d^t, z^t, q^t) P^*(d^t, q^t, z^t | z_0, q_0)$$

The price system is suggested by the risk neutrality of the banks and entrepreneurs. But, this price system is valid *if and only if* the credit constraints insure that active entrepreneur will not default. Obviously, the value of a security that pays in the states of the world in which the entrepreneur defaults must be equal to zero.

The price system indicates that the liabilities of dead entrepreneurs or liabilities of entrepreneurs that payoff only in the states where the entrepreneur dies must necessarily be zero. The valuations of these securities do not pose any discussion. But notice that we have specified that the liabilities of agents who have been previously liquidated have zero value too. This obeys

to the assumption held all over the paper that once the agent is in the underground sector, he does not have any incentives to repay debts. Notice also that since we are assuming  $q \geq \beta$ , liquidated agents would not want to save, and therefore, the assumption is not with loss of generality.

From the long term contract, the obvious solvency constraints are:

$$A_t(d^t, z^t, q^t) = \begin{cases} -K_0 & \text{if } t = 0 \text{ and } V_0(z_0, q_t) > U \\ C(z_t, q_t, G_{z_t, q_t}(z^{t-1}, q^{t-1}, V_{t-1}(z^{t-1}, q^{t-1}))) & \text{if } d_s = 1 \text{ all } s \leq t \\ & \text{and for all } s < t \\ & G_{z_s, q_s}(z^{s-1}, q^{s-1}, V_{s-1}(z^{s-1}, q^{s-1})) > U \\ -L & \text{if } d_1 = \dots = d_t = 1 \text{ and} \\ & G_{z_t, q_t}(z^{t-1}, q^{t-1}, V_{t-1}(z^{t-1}, q^{t-1})) = U \\ 0 & \text{otherwise} \end{cases}$$

while the simplest specification of borrowing constraints are given by

$$B_t(d^t, z^t, q^t) = \begin{cases} k^\sigma(z_t, q_t, V_t(d^t, z^t, q^t)) & \text{if for all } s \leq t, d_s = 1 \text{ and} \\ & G_{z_s, q_s}(z^{s-1}, q^{s-1}, V_{s-1}(z^{s-1}, q^{s-1})) > U \\ 0 & \text{otherwise} \end{cases}$$

The  $\mathcal{A}, \mathcal{B}$  profile is very simple and intuitive. First, the solvency constraints allows the entrepreneurs in their first period of life to borrow up to the cost of the plant  $K_0$  *only if*, given the economywide interest rate and the realized idiosyncratic productivity  $z$  the entrepreneur can expect a utility higher than  $U$ . The solvency constraints specify that in those states of the world in which the entrepreneur is going to be liquidated, then he can owe at most the liquidation value of the plant. Entrepreneurs that have been liquidated or died previously cannot negative balances.

Borrowing constraints are necessary because the entrepreneur might be better off by appropriating the working capital at hand and leaving the financial system with all the liabilities unpaid. For active agents the constraint is the incentive compatible upper limit that results from comparing the utility that a maximizing agent can achieve inside the system to the one attained by defaulting. Trivially, agents dead in that period cannot borrow, but more importantly, entrepreneurs who are alive but that have been previously or are currently being liquidated, cannot

borrow in positive amounts, as they would not have the incentives to repay.

With the candidate  $\mathcal{A}, \mathcal{B}, p^*$  in the place, the argument is exactly as for the case of certainty. First, by construction the allocation  $\sigma$  of the infinite horizon contract is feasible. Second, given the constraints, the allocation is indeed optimal. To see this, notice that constraints and prices given,  $\mathcal{P}(\mathcal{A}, \mathcal{B}, p^*)$  is a convex problem. Since  $\beta < q$  the transversality condition holds trivially, the Euler conditions are sufficient. Given prices  $p^*$ , we can readily verify that the allocation  $\sigma$  satisfy the Euler equations. The details are in the appendix. Obviously, this implies that the entry decisions coincide in both environments.

To finish the argument, we then consider  $\mathcal{P}(\mathcal{A}, \mathcal{B}, p^*)$  allowing the agent to default and to decide whether to participate. By our assumptions, once the entrepreneur defaults or is liquidated, his problem becomes trivial: consume  $e$  every period. This is a logical implication of the assumption that once in the background sector, he cannot be forced to honor any liability.<sup>20</sup> The profile  $\mathcal{A}, \mathcal{B}$  allows trading strategies that involves nodes in which the entrepreneur will be better off defaulting. Some of those trading strategies require that the agent consume over time **below** the dictates of  $\sigma$  and accumulate assets  $a$  above the specification of  $\mathcal{A}$ , or that accumulates assets. Since  $q < \beta$ , there are values  $a$  high enough so that the entrepreneur would find it optimal to borrow  $B_t$  and default. By construction, the incentive problems would arise in excessive savings as the solvency constraints impel the agent to save. However, once the agent is in the path of  $\sigma$  we would find it suboptimal to implement any of this strategies. Optimizing entrepreneurs will start and remain in the path  $\sigma$ .

Conditional on  $z, q$ , there is a one-to-one positive relationship between the state  $V$  in the economy with long term contracts with the equilibrium asset  $a$  owned by the entrepreneur in the centralized asset trading environment. Both can legitimately be thought of as collateral: the first one because of the full commitment of the bank to honor his commitments; the second for much more obvious reasons. We refer to them interchangeably as collateral.

## 6 Which Firms are Credit Constrained?

In this section we discuss the credit constraints implied the limited commitment of the entrepreneurs. In our environment, credit limits can affect firm behavior in the scale of operations (intensive margin) as well as in the entry and exit (extensive margins). Definitions of these notions of credit constraints arise naturally after we compare the allocations with those in an environment with full enforcement. We then link the likelihood of a firm to be constrained in

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<sup>20</sup>Otherwise, given that  $\beta < Q$  the entrepreneur would want to borrow everything,  $e/(1 - q)$  and consume it right away. Of course, the next period he would default and consume his endowment.

terms of its age and size.

In a world of perfect commitment the no-default constraint disappears. Once they have signed the contract, entrepreneurs cannot opt out. With no default constraint disappear, regardless of their age and history, active firms, would always use the unrestricted level of working capital  $k^u(z, q)$  and the period surplus is equal to the unconstrained profits  $\pi(z, q)$ . The most obvious notion of inefficiency is that active firms cannot use the optimal level because they cannot commit not to default:

**Definition 1.** *Given  $(V, z, q)$  a firm is **credit rationed in the intensive margin** if remains active but  $k(V, z, q) < k^u(z, q)$*

But credit constraints can also affect entry and exit decisions. To see this, we need to lay out the value function for ongoing relationships need only to satisfy the promise keeping constraint and the non-negativity of entrepreneur's consumption. Therefore, the set of feasible promise utilities  $\Gamma^u$  is given by

$$\Gamma^u(V, z, q) = \left\{ y : Z \times Q \rightarrow [0, +\infty) \text{ s.t. } \sum_{z', q'} y(z', q') P(z', q' | z, q) \leq \frac{V}{\beta(1 - \delta)} \right\} \quad (27)$$

It is not hard to see that since  $q \geq \beta$ , the full enforcement allocations requires that after the initial period, as and as long as the firm is in operation, the consumption of the entrepreneur must be equal to zero. Entrepreneurs receive transfers from the bank in the first period and they consume all it right away.<sup>21</sup> Using this feature, the value function for ongoing relationships is simply:

$$C^u(z, q) = \min \left\{ -L, \left\{ -\pi(z, q) + q(1 - \delta) \sum_{z', q'} C^u(z', q') P(z', q' | z, q) \right\} \right\} \quad (28)$$

Of course, there is no need for randomizations on  $V$ . The initialization of contractual relationships is done a similar fashion as above. Because  $\pi(z, q) \geq S(V, z, q)$  and  $\Gamma(V) \subset \Gamma^u(V)$ ,  $\forall V \geq U$ , it is obvious that  $C^u(z, q) \leq C(V, z, q)$  all  $(V, z, q)$ . With commitment, entrepreneurs would receive a higher initial entitlement. The second type of inefficiency arises because some firms that would be created with perfect enforcement are not activated because incentive problems along the expected operation of the plan. This is

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<sup>21</sup>The contract under full enforcement resembles the very sad situation described by Horacio Quiroga in *Los Mensú*.

**Definition 2.** *Given  $(z, q)$  a firm is **credit rationed in the (extensive) creation margin** if simultaneously*

$$\exists y^u \geq U \text{ s.t. } y^u - U + C^u(z, q) + K_0 \leq 0 \quad \text{and} \quad \forall y \geq U, C(y, z, q) + K_0 > 0 \quad (29)$$

Similarly, for subsets of exogenous states  $Z \times Q$ , some firms might be destructed only because of the enforceability problem. This type of credit rationing can be defined as:

**Definition 3.** *Given  $(V, z, q)$  a firm is **credit rationed in the (extensive) destruction margin** if*

$$C(V, z, q) = V - U - L > C^u(V, z, q)$$

All three types of credit rationing can be present, depending on the processes  $z$  and  $q$  and the values of  $\alpha, K_0, L$  and  $U$ . These notions of rationing are not independent. Rationing in the intensive margin is a necessary but not sufficient condition for the existence of rationing in any of the extensive margins.

Which firms are more likely to be constrained? The larger is the value of  $V(a)$ , the larger is the amount of working capital that banks can advance to the entrepreneur without concerns of triggering his default. Thus, conditional on  $z, q$ , firms who poses a larger collateral in the initial period are less constrained. In economies in which the limited liability constraint is binding in the first stages of life of the firm, then younger firms are more likely to have a lower level of  $V(a)$  than more mature firms. Moreover, again conditional on  $z, q$ , for a firm to be larger than other is just equivalent to be less constrained. Now, since the optimal usage of working capital for more productive firms is also higher, one could suspect that they are more likely to be constrained in the intensive margin. But this is not necessarily the case because the optimal policy function specifies that  $G_{z',q'}(\cdot)$  is strictly increasing in  $z'$ .

Firms with small levels of collateral are also more likely to be constrained in the destruction margin. Remember that the optimal liquidation policy requires no randomization and that when firms are liquidated, all the assets are seized. The more binding the limited liability constraint, i.e. the lower  $V(a)$ , the larger is the set of states in which the next period the firm will be liquidated. Clearly, the lower the productivity, the more likely those firms will be shut down. The higher the productivity of the firm, the less relevant the participation constraint becomes because the less attractive is the underground sector.<sup>22</sup> Firms with lower productivities are

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<sup>22</sup>Note the difference with the endowment economies of Kehoe-Levine and Alvarez-Jermann [39, 2]. In their case, in autarky the entrepreneur retains his endowment, and therefore higher current productivities make the participation constraints more binding. In our case, the autarky is independent of the productivity  $z$  which of the technology, which can only be used if the entrepreneur remains in the financial system.

also more prompted to be constrained in the creation margin.

Higher interest rates are associated with more firms constrained in the extensive margins. In our setting the continuation value of firms depends on interest rates while the scrapping value  $L + U$  does not. In the regions where liquidation/creation decisions are made, the value of continuation is decreasing in the interest rate. Therefore, with higher interest rates less firms will have a continuation value above  $L + U$  for continuation of  $K_0 + U$  for creation. Just because of the limited commitment of the entrepreneur, more firms are destroyed and less are created.

## 7 Asymmetries in the Effect of $q$ Shocks

The realization of  $q$  affects not only the pool of new entrants, but also which of the active firms are liquidated. For entry, there are two opposite effects: higher interest implies a higher initial  $z_0$  for a firm to be active; but given  $z_0$  the newly created firms are more credit constrained as the value  $V_0(z_0, q)$  is decreasing in  $q$ . It also turns out that the effect on active firms depends on their value  $V$ . The liquidation of firms with smaller  $V$  is more responsive to  $q$ . This section extends this discussion.

### 7.1 Asymmetric Effect of $q$ across active firms

The function  $C$  is not monotone in  $q$ . In this model,  $q$  shocks are relevant only for their effects on  $Cc$ , the continuation value as the scrapping value of the plant is independent of  $q$ . On one hand, the timing between the commitment of working capital and the recollection of output implies that with higher interest rates, the net present value of the profits of the technology is reduced. But on the other hand,  $q$  discounts the present value of the consumption (dividends) for the entrepreneur, and hence higher  $q$  implies *lower*  $C$ ; the net effect results of the balance of these two forces. Because the state  $V$  has the dual role of determining both profits and consumption streams,  $C$  is not submodular in  $(V, q)$ . But for larger values of  $V$ , the imperfect enforceability is not binding and the effect on the present value of consumption dominates. Imperfect enforceability implies that, for same productivity  $z$ , firms with lower  $V$  (younger and smaller) are more likely to liquidate in periods of high interest rates.<sup>23</sup>

Even if in equilibrium the contract never arrives to regions in which  $C$  is increasing in  $q$  (increasing in the real interest rate), interest rates affect more the continuation of small firms,

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<sup>23</sup>Perhaps the intuition is more clear using the decentralize trading environment. Interest rates reduce the present value of the firms. But there is also a wealth effect of  $q$ : if  $a < 0$  higher  $q$  reduces further the expected present value of the consumption for the agent that continues in the financial system. If  $a > 0$  the effect is the opposite and if  $a$  is high enough, the expected utility of the agent can actually increase with higher interest rates.

because of the transition dynamics. Firms that have accumulated enough collateral can use more working capital and the total expected discounted value of the surplus from the technology can dominate the opportunity cost  $L + U$ , and the plant continues. However, during the first periods the firm is operating at a much lower scale, and the present value of the surplus of the firm might fall below the liquidation value.

To illustrate this assume that  $z_t = z$ ,  $q_t = q$  for all  $t$ , and  $\delta = 0$ . Here, a relationship either liquidates immediately or remains active forever; continuing relationships converges to a steady state. During the transition,  $V_t$  grows as fast as possible until reaching its steady state value. Let  $M(q)$  and  $S(q)$  denote respectively, the steady state level of  $V$  and the period profits extracted from the technology in steady state. Let  $V_0 \leq M(q)$ . Obviously the cost  $Cc(V_0, q)$  for a bank to maintain a relationship with initial value  $V_0$  equals the net present value of the profits (NPVP) minus the NPV of the dividends (NPVC), or

$$Cc(V_0, q) = NPVC(V_0, q) - NPVP(V_0, q) \quad (30)$$

Defining  $T^*(V_0, q)$  the number of periods needed to achieve the steady state, i.e.

$$T^*(V_0, q) = \min_{t \in \mathbb{Z}} \{ \beta^{-t} V_0 \geq M(q) \} \quad (31)$$

we can decompose the previous terms in steady state and transition components:

$$NPVP(V_0) = \sum_{t=0}^{T^*(V_0, q)-1} q^t S(V_0 \beta^t) + q^{T^*(V_0, q)} \frac{S(q)}{1-q} \quad (32)$$

$$NPVC(V_0) = q^{T^*(V_0, q)-1} [q(\beta^{T^*(V_0, q)-1} V_0 - \beta M(q))]^+ + q^{T^*(V_0, q)} \left[ \frac{qM(q)(1-\beta)}{1-q} \right] \quad (33)$$

Notice that  $T^*(V_0, q)$  is strictly decreasing in  $V_0$ , and because for all  $t \leq T^*(V_0, q)$   $S(V_0 \beta^t) < S(q)$  the  $NPVP(V_0)$  is strictly decreasing in  $V_0$ .

If,  $\frac{S(q)}{1-q} > L + U$  then those firms that have achieved the steady state remain active at the given rate  $q$ . Similarly for firms that are close to the steady state. However, it may easily be the case that for a low value  $V_0$   $NPVP(V_0)$  falls way below  $L + U$ , and they are optimally liquidated.

This is reinforced by the effect on the NPV of the dividends. One can verify that  $V_0 - NPVC(V_0)$  is strictly increasing in  $V_0$

The previous analysis extends to the general case. [write briefly]

## 7.2 Effects on Entry: Cleansing vs. Sullyng Effects

Is the average productivity of the resources used by the plant higher or lower with higher interest rates? In general, higher interest rates (lower  $q$ ), heighten the inefficiencies of the timing of production. Therefore, the working capital used by all active plants is reduced, implying, that for all  $z$  the marginal product of working capital is higher and therefore its average productivity also higher. This *intensive margin* effect is reinforced by the *extensive margins* because surviving and newly created firms have also higher  $z$ s. Therefore, if we are only concerned with the productivity of the working capital used by the firms, productivity is higher with higher interest rates (recessions). Moreover, extensive margin responses induce *cleansing* in the pool of firms in terms of  $z$ .

On the other hand, if we also consider the average productivity of the fixed capital, which in the model is given by *plants*, then it is very likely that the average productivity declines, depending on the relative strengths of the intensive versus the extensive margins effects, which now operate in the opposite direction. Moreover, there is a *sullyng effect* in the sense that with higher interest rates, the entrants are more credit constrained, a fact that reduces the productivity of the plant as well as the chances of future survival.<sup>24</sup>

## 8 Aggregate Dynamics

Because the market discount factor is a common component in the state of every ongoing relationship, it is only necessary to track  $q_t$  and the distribution of the idiosyncratic components of the active relationships. Hence, the state for the whole economy is the pair  $(q_t, \psi_t)$  where  $\psi_t$  is a measure over  $(V, z)$  the idiosyncratic components of the state of individual relationships.

The state space for  $\psi_t$  cannot be taken to be a space of probability measures because the total mass of active firms varies according to the entry and exit. However, the relevant measures are bounded because at most the entire unit mass of agents can be active. Let  $\mathbf{Y} \equiv [U, +\infty) \times Z$ , the set of idiosyncratic states and  $(\mathbf{Y}, \mathcal{Y})$ , the respective measurable product space. Denote  $M_1 \equiv \{\psi \in M(\mathbf{Y}, \mathcal{Y}), \psi(\mathbf{Y}) \leq 1\}$ , and  $\Lambda \equiv \{\psi \in M(\mathbf{Y}, \mathcal{Y}), \psi(\mathbf{Y}) = 1\}$  the set of measures bounded by the total mass of living agents and the set of probability measures.

The contracts defined at the individual level will define operators that render the aggregate time series of creation,  $(T_C)$ , destruction of firms,  $(T_L)$ , as well as the evolution of the states for the enduring firms,  $(T_E)$ . Respectively, these operators take the form of the mappings,

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<sup>24</sup>Gadi Barlevy pioneered the exploration of sullyng effects.

$$T_C : Q \rightarrow \Lambda \quad (34)$$

$$T_L : Q \times Q \times M_1 \rightarrow [0, 1] \quad (35)$$

$$T_E : Q \times Q \times M_1 \rightarrow M_1 \quad (36)$$

Since  $\{q\}$  is taken to be exogenous, we need only to trace the dynamics of  $\{\psi_t\}$ . For simplicity the exposition omits the randomizations in the exit margin. Given the interest rate of the period,  $T_C$  will provide us with a measure over the entrant entrepreneurs. Notice that I have assumed that entry decisions are based on the productivity of the current period and that these productivities are drawn from the unique invariant c.d.f.  $F_z$ . Either all, none or only a fraction of the possible entrants become active and that will depend on the realization  $q$ .

Competition fixes the initial value  $V_0$ . Those agents who are not initialized obtained a promised utility entitlement of  $U$ . If  $V^0(z, q) > U$  the agent became active. It's easy to keep track of the population of entrants. For any pair of Borelian sets  $A \times B \subset (U, \infty) \times Z$ ,  $T_C(q_t)(A \times B)$  is given by

$$T_C(q_t)(A \times B) = \sum_{z \in B} \chi[V^0(z, q) \in A] F_z(z) \quad (37)$$

and therefore the mass of entrants in  $A \times B$  is

$$\psi^0(q_t)(A \times B) = \delta T_C(q_t)(A \times B) \quad (38)$$

We adopted the convention to denote by  $G_{z', q'}(V, z, q) = U$  the case when a firm is liquidated. The total mass of firms begin liquidated, when the previous state was  $(q_{t-1}, \psi_{t-1})$  and current exogenous state is  $q_t$  is given by

$$T_L(q_t, q_{t-1})\psi_{t-1} = \int_{R_U \times Z} \left( \sum_{z \in Z} \chi[G_{z', q_t}(V, z, q_{t-1}) = U] P_z(z', z) \right) \psi_{t-1}(dV \times dz) \quad (39)$$

Finally, enduring relationships  $G_{z', q_t}(V, z, q_{t-1}) > U$  will update their state so that for any  $A \in \mathcal{B}(U, \infty)$  and  $B \subset Z$

$$T_E(q_t, q_{t-1})\psi_{t-1}(A \times B) = \int_{R_U \times Z} \left( \sum_{z \in B} \chi[G_{z', q_t}(V, z, q_{t-1}) \in A] P_z(z', z) \right) \psi_{t-1}(dV \times dz) \quad (40)$$

The exposition has ignored so far the exogenous death of firms. Since only  $(1 - \delta)$  firms actually overcome the exogenous risk the transition of  $\psi$  is actually given by

$$\psi_t = (1 - \delta) [T_E(q_t, q_{t-1})\psi_{t-1}] + \delta T_C(q_t) \quad (41)$$

## 8.1 Stationarity of The Aggregate Economy

Because of the presence of aggregate uncertainty, in general these economies cannot converge to an invariant distribution of firms.<sup>25</sup> However, there is a sense in which their limiting behavior is stationary, i.e. independent of calendar time.

Rolling backwards the transition for  $\psi_t$ , for any positive integer  $N$

$$\psi_t = \delta T_C(q_t, q_{t-1}) + \delta \sum_{j=1}^{N-1} (1 - \delta)^j P_E(q_t : q_{t-j}) T_C(q_{t-(j+1)}) + (1 - \delta)^N P_E(q_t : q_{t-N}) \psi_{t-N} \quad (42)$$

$P_E(q_t : q_{t-j}) : M \rightarrow M$  of the form

$$P_E(q_t : q_{t-j}) \equiv T_E(q_t, q_{t-1}) T_E(q_{t-1}, q_{t-2}) \dots T_E(q_{t-j}, q_{t-j-1}), \quad (43)$$

for  $j = 0, 1, 2, \dots$ . Because  $T_E$  is mass decreasing so is  $P_E(q_t : q_{t-j})$ , and because  $(1 - \delta)^N \rightarrow 0$  as  $N$  grows, the effect of  $\psi_{t-N}$  on  $\psi_t$  washes out over time. Equivalently, the effect of  $\psi_t$  on  $\psi_{t-N}$  vanishes as  $N \rightarrow \infty$ . Taking that limit

$$\psi_t = \delta T_C(q_t) + \delta \sum_{j=1}^{\infty} (1 - \delta)^j P_E(q_t : q_{t-j}) T_C(q_{t-(j+1)}) \quad (44)$$

Because  $\{q_t\}$  is a stationary process and  $T_C, T_E$  are time invariant operators, it is immediate that the distribution of  $\psi_t$  does not depend on the calendar time. This is very convenient because all the time series that we are interested in are functions of  $\psi_t, q_t$ , and as such, they will also be stationary.

A special case is when there is not aggregate uncertainty, i.e., when  $q_t = q \in (0, 1)$  for all  $t$ . In this case, independently of the initial conditions, the economy will converge to a unique invariant distribution. To see this, notice that the measure of entrants  $T_C(q)$  is time invariant and that  $P_E(q_t : q_{t-j}) = T_E^j(q, q)$ ,  $j = 0, 1, 2, 3, \dots$ . Then the unique invariant distribution is

<sup>25</sup>A trivial exception is when no firms are ever created.

$$\psi(q) = \delta \left[ \sum_{j=0}^{\infty} (1 - \delta)^j T_E^j(q, q) \right] T_C(q) \quad (45)$$

## 8.2 Numerical Illustrations

In this section we presents a brief review of the features of the aggregate fluctuations in the artificial series generated by the model. We believe that the model abstracts from important shocks and transmission mechanisms driving the aggregate fluctuations in actual economies. The exercises summarized here are aimed to show the qualitative features, and, at this stage, we are not aiming to match the moments of the data.

Notice, anyway, that a complete calibration exercise is not entirely feasible as many objects in the model do not have an obvious empirical counterpart, for example the set-up cost  $K_0$ , the liquidation value of the plant  $L$  or the reservation utility  $U$ . Other ingredients of the model, for example,  $z$  the stochastic process of the idiosyncratic productivity of plants, are hard to measure, specially if one must consider credit constraints of the firms as well as other elements (e.g. learning) ignored in the model.

However, after running simulations for many parameter specifications, the qualitative strengths and weaknesses of the model are very robust to the parameter. We report here the case for parameter values we think are reasonable. The table presents the parameter specifications. We also specify  $z$  given the common belief that idiosyncratic productivity shocks are very persistent. We set  $z$  as a Tauchen-Hussey Markov chain approximation of a log normal process, with a correlation of 0.9. The support  $Z$  was normalized to be contained in an close interval.

For the process follow by the interest rate, we also took the Tauchen-Hussey Markov chain approximation of the actual quarterly FedFunds rate. In some periods, the ex-post real interest rate is negative, and we re-scale it so that its support lies within  $[\beta, 1)$ . Therefore, the main ingredient for the aggregate fluctuations is directly taken from the data. In the section C of the appendix we display the figure of the actual interest rate process with the Markov chain approximation. The other parameters are displayed in the table.

A problem with doing the simulations is the discreteness of shocks. We need to use numerous supports  $Z$  and  $Q$ , because with a small number, the  $z$  shocks have radical effects on the survival of firms and the  $q$  shocks radical effects on the creation and destruction series. We use  $N_z = N_q = 5$  number of states for each shock. This allows us to generate series that are not too extreme and keep the computations feasible. Yet, notice that we have to solve for the optimal contract we need to solve a dynamic programming problem with 25 states  $(z, q)$ . The Bellman Equation is solve using quadrature based methods, as discussed in Christiano and Fisher [14].

Table 1: Parameter Values Used

Parameter	Value Used
Discount Factor Entrepreneurs	$\beta = (1.1)^4$
Death Prob./Entry Mass	$\delta = 0.025$
Entrepreneur's backyard income	$e = 4.79, \Rightarrow U = 100$
Liquidation Value Plants	$L = 30$
Installation Cost of Plants	$K_0 = 33.3333 = L/0.9$
Output elasticity to working capital	$\alpha = 0.85$
<b>Shocks</b>	
Idiosyncratic Productivities $z$	$Z \subset [1.75, 2.25], N_z = 5$ $\rho_z = 0.9, \sigma_z = 0.12$
Market Discount Factor $q$	$Q \subset [\beta, 0.99999], N_q = 5$ $\rho_q = 0.62, \sigma_q = 0.00635$

In the section C of the appendix we discuss in detail the algorithm used. There we also display the form of the value function  $C$  and of the policy function  $G$ .

Not surprisingly, the model is to the values assigned to  $L, K_0, U$ . It is not hard to find specifications of these parameters such that there is no entry at all or that all the firms enter and never exit. Also, for very low values of  $U$  the incentives are not binding, and the scale of operations of firms is independent of the age. It can be time consuming to find parameter specifications where the allocations are interesting. Notice that the model is very *tight* in parameters. The only real *free* parameter in the model is  $K_0$  that governs entry. Any other parameter affects the fixed point  $C$ , and hence entry, exit, growth, etc. in non-trivial ways.

Much more interestingly, the results are very sensitive to the elasticity of the production function with respect to working capital. The closer  $\alpha$  is to one, the closer to linear is the production function and the larger is the optimal unrestricted working capital. In those cases, profits, entry and exit are very sensitive to the shocks. Given that  $\beta, \delta$  have to be set to realistic values, they do not allow for much manipulation,  $\alpha$  is very much the only room for altering the number of periods to maturity of the firms. Thus, the larger  $\alpha$  the longer is the life cycle dynamics. Because of computational limitations, one cannot push too much because then the set  $[U, M(\bar{z}, \bar{q})]$  becomes arbitrarily large, and we would need many more grid points to have an accurate approximation.

Whenever the model is set with parameter specifications in which interest affect entry and exit, the model implies comovement of aggregate flows with interest rates, but the environment

also allows for either margin to respond more strongly. For instance, if  $K_0$  is too large with respect to  $L, U$  then most of the response to interest rates is in the creation margin. This is, once a firm is created, it is not liquidated because the liquidation value is insignificant. On the other hand, since  $K_0 > L$ , it is a necessary condition for entry to respond to interest rates that exit respond to them. Yet, it is not a sufficient condition. It is not hard to set the shocks  $Z$ , specially when working with a small number of them, so that a constant fraction of firms enter. All these possibilities are not really interesting.

But, the qualitative results in environments where interest rates affect *both* entry and exit, are very robust. As expected, both creation and destruction of firms are very strongly correlated with the interest rates. From the previous paragraph, we know that the relative magnitudes are not of direct interest because they are driven by assumptions on parameters that we have not estimated and do not have the information to calibrate. Figure 6 displays the spectral densities of the artificially generated flows of firms. Several patterns in the figure are interesting and robust. First, the model implies a much more volatile series of destruction than of creation. Second, and more interestingly, most of the extra volatility is in higher frequencies. The largest fraction of the variability of firm liquidation is done in very short periods of time. The artificial destruction series are obtained by feeding the model with interest rates with a large fraction of the variation in low frequencies, and therefore the model implies that after a long live innovation of interest rates, the firms that survive are likely to remain active in the next period.

Therefore, the destruction series are highly temporally concentrated in the sense discussed above. Notice that this is true either if we look at the rate (mass of flow/mass of active) or the mass of the destruction. Comparing Figure 3 with Figure 6 we observe that in general, the model predicts that the process of destruction should be much more temporally concentrated than in the data. Also, observe that given the behavior of entry, the result goes through to net growth of active firms.

Looking at the behavior of firm creation, indicates a limitation of the modeling of entry in the model. While the implied spectra implies that the relative importance of the different frequencies are similar to the implied by the data in Figure 3, the result holds mostly because of the tight connection between entry and interest rates, which is much higher than the one observed in the data. In the conclusions we discuss how to extend the modeling of the entry even within the context of an OLG framework used here.

The information on the third panel of the Figure 6 is very important for the implied dynamics of output. As we can see, the stochastic process follow by the mass (number) of active firms, while showing an important degree of temporal concentration, its variability is also strongly governed by low frequency innovations.

Figure 6: Model's implied Spectra of Firm Flows

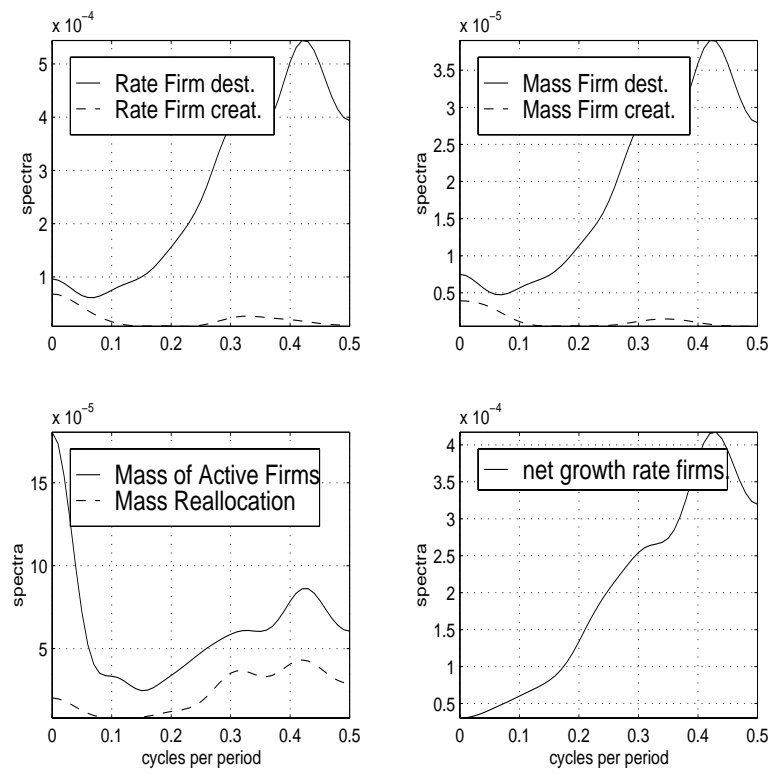


Figure 7: Spectra for Aggregate Output: Model and U.S. economy

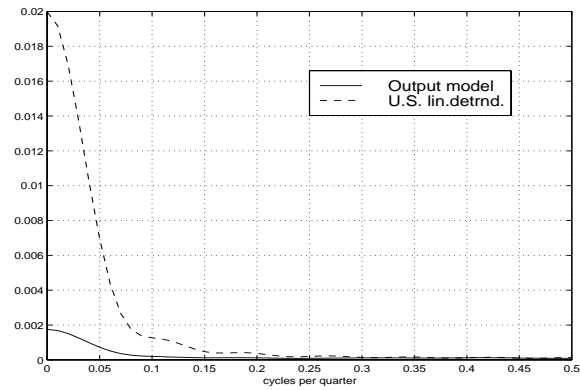


Figure 8: Model's Implied Real Interest Rate/Aggregate Output Correlations

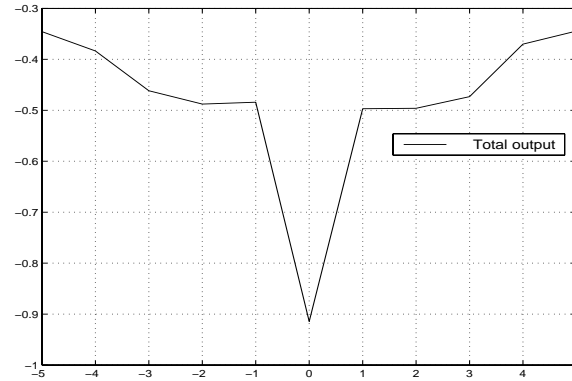
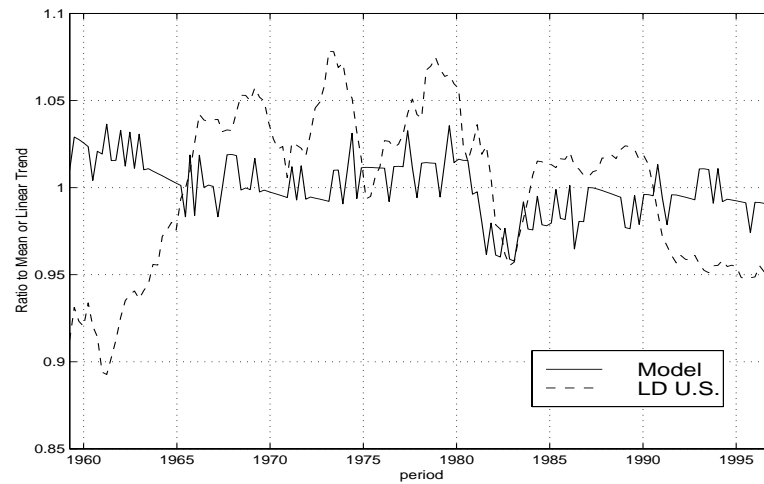


Figure 9: Aggregate Output Realizations: U.S. and Model Economy



The persistence in the series of active firms goes through to the implied series of output. Figure 7, displays the spectra of actual U.S. linearly detrended output and the implied by the model. (Below we explain how we make the series comparable.) The implied output series have important persistent components, as can be seen from the relative importance of low frequencies. Indeed, the output series are more persistent than the driving force. While first order autocorrelation of quarterly real gross interest rates 0.62 the implied autocorrelation of output in this example is 0.81 (output defined as net profits  $z_t f(k_t) - k_t$  of active firms plus the income  $e$  of agents in the background; using only the first component increase the autocorrelation to 0.85). While the numbers can be made even larger, I have not run a case that matches the autocorrelation of actual output, which is 0.9775.

The persistence of aggregate output is largely delivered by the persistence in the mass of active firms, which unfortunately is not empirically observable. For reasons discussed in the conclusions, the life-cycle dynamics of firms does not provide much additional persistence. Figure 8 shows the dynamic correlations between artificial aggregate output and real interest rates. As in the empirical evidence, the association is negative. Interestingly, except for the zero lag, the correlations are very similar to the ones obtained actual linearly detrended output. But the aggregate output in the model is much more responsive to interest rate than documented by the data.

The limitations of the model are more evident by looking at the series of output generated by the model once the *actual realization* of interest rates are fed into the model. Figure 9 shows the data for the U.S. and the implied by the model. The U.S. output series are expressed as a ratio to the trend, while the artificial series as a ratio of the sample mean. The artificial data are started with the invariant measure of firms associated with the median  $q = q_3$  interest rate. By the radical mismatch in the first periods, it is clear that we are not giving the model its best shot. It is interesting to observed that the behavior of the model is in line with the data in some periods where the U.S. authorities were known to tighten the monetary policy. Moreover, the series show some persistence, which, as we said below is driven by the measure of active firms. However, the artificial series is consistent with our observation that the model predicts that the adjustments in the economy are more concentrated in short periods of time. Clearly, the data displays much more persistence. And of course, the effect of shocks omitted in the model.

## 9 Concluding Remarks

We studied a model in which the creation, expansion and liquidation of firms are determined and limited by incentives to default of the entrepreneurs. We studied the allocations from optimal

long-lived financial arrangements and derived the implied firm dynamics. We generalized the work of Albuquerque-Hopenhayn [1] on individual firm dynamics by allowing shocks in the interest rate faced by the bank. This extension is used to construct an equilibrium model and study the implications of interest rate fluctuations on the aggregate flows of creation and liquidation of firms, the reallocation flows across active firms and the aggregate output.

But, despite our main focus on aggregate dynamics, we extended the micro analysis of the long term contract under imperfect enforceability. We provided an alternative trading arrangement, one in which entrepreneurs finance their firms using one period securities in an environment with centralized information, that can replicate the allocations attained in the infinite horizon one-side commitment environment. The exercise provides an explicit link of the firm behavior with its collateral. We also verify that even so the value function is not necessarily monotone, the equilibrium allocations are always renegotiation proof. More interestingly, we also establish that randomizations are necessary to convexify the problem, deterministic allocation rules are sufficient as non-trivial randomizations are only relevant outside the equilibrium allocations.

The dynamics of aggregate output requires explicit consideration of the cross-section of active firms. The model implies interesting effects of interest rate shocks as they affect firms asymmetrically depending on their collateral. Indeed, the larger the collateral of firms, the less sensitive is the exit probability. Therefore, the survival of small, younger firms is more sensitive to interest rates than that of larger, more mature firms. The model has also interesting implications on entry. On the one hand, with higher interest rates less firms are created, and the minimum productivity needed for activation is higher. Therefore, higher interest rates purify the pool of entrants and therefore their survival probabilities must be higher. But, on the other hand, conditional on their idiosyncratic characteristics, the firms that are created during recessions face tighter credit constraints, their operations are more limited and their survival probabilities are lower.

With the approximate quarter real interest rates of the U.S. economy, we feed a parameterized version of the model and examine the artificial series. Implied aggregate output and job and firm flows have several qualitative features found in the data. Real interest rates comove negatively with output, gross creation, net growth, of jobs and firms, and negatively with flows of job and firm destruction. With its built-in aggregate frictions in the creation of firms, the model also compatible with the volatility of destruction flows being much larger than that of creation flows. Furthermore, the response in the mass of active firms propagate the interest rate shocks, as the resulting persistence of output series is higher than that of the original interest rate shocks. Additionally, the destruction series are highly concentrated in short periods of times.

We have not subjected the model to formal empirical testing. Yet, the numerical simulations

described at the end of the paper suffice to grasp important qualitative limitations of the model. First, output and gross firm and job flows comove much more tightly with the interest rate in the model than in the data. Second, with only working capital, the life cycle dynamics of firms is rather short, which reduces the potential of endogenous persistence in the model. Finally, comparing the model series with the evidence on the manufacturing data in the U.S. makes it clear the serious limitations in our modeling of entry. We conclude the paper by discussing the extensions of the model to cope with the previous limitations.

The general equilibrium embedding of the model, which is discussed in the appendix, is designed to abstract from the feedback of the measure of active firms on the determination of the interest rate. Such obvious limitation of the analysis is made for analytical tractability, which is badly needed in the context of infinite horizon contracts. The modeling strategy has the advantage of focusing directly on interest rate shocks in addition of abstracting from the particular details that drives the shock in  $q$ . But, while some of the implied fluctuations are in line with qualitative features of the actual data, it is very clear that the model leaves aside important sources of the fluctuations in actual economies, and over-emphasize the sole aggregate shock. An extension, available to any other model, is to complicate the set up by including other, say technology, shocks, as in Cooley-Quadrini [16]. A more interesting route would be to allow for another input, as in Bergin-Bernhardt and Caballero-Hammour [3, 11], whose price, say  $w$ , must clear the markets. We have reasons to believe that the asymmetric effect across firms and the direction of effects on aggregate output of shocks in the assets markets will be robust to those extensions. The general equilibrium effects would only affect the timing of the response of the aggregate economy and will most likely deepen the asymmetry in the effects across firms.

The overlapping generations structure imposed on the model effectively operates as an aggregate friction in the creation of firms and jobs. In this sense, the model incorporates frictions stressed by the search and matching literature or the more barefaced aggregate creation costs, e.g.[10].<sup>26</sup> With this structure in hand, the model easily produces destruction flows that are more volatile than creation, aggregate persistence due to sluggish response of the mass of firms, and asymmetry of the recessions versus expansions.

However, the model's creation series show the shortcomings of delivering the results from this assumption. Counterfactually, creation inherits practically all the properties of the interest rates. An extension to push in the near future, is to assume that the entrepreneur once liquidated can return. Every period the agent is in the underground sector he draws a productivity from the invariant distribution  $F_z$ , independent of the previous entrepreneurial spells. At the aggregate economy, there is a measure  $\psi_t$  of active entrepreneurs and a mass  $n_t$  of agents in

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<sup>26</sup>Notice that the OLG structure we employ poses much more discipline in the sense of fewer parameters.

underground. The pool of entrants is described by  $n_t F_z$  and depends on the state  $q$ . With this added feature, the reservation utility,  $U^*(q)$  will be a function of the current interest rate, affecting exit decisions. Entry decisions must now solve an optimal stopping time problem, making entry more sensitive and, we speculate, more temporally concentrated.

Finally, an extension model that will enhance persistence and the asymmetries of expansions and recessions is to include physical capital accumulation in the technology managed by firms. If the production function is  $f(K, k)$ , and in addition of the working capital the entrepreneurs could appropriate a fraction  $\theta K$  of the physical capital under their control, the life cycle of firms will be much longer. Here, the value of the entrepreneur  $V$  and the capital  $K$  under his control will be complementary, and before the  $K$  can be expanded without triggering default, the value  $V$  must be increased, a time consuming process because of the limited liability of entrepreneurs.

# Appendices

## A Proofs

### A.1 Preliminaries and Notation

Let  $R_U \equiv [U, \infty)$ ,  $F \equiv \{f : R_U \times Z \times Q \rightarrow R, f \text{ bounded and continuous in the first argument}\}$ , and the norm  $\|f\| = \max_{(z,q) \in Z \times Q} \sup_{x \in R_U} |f(x, z, q)|$ . Obviously,  $(F, \|\cdot\|)$  is a normed linear space. With the metric  $d(f, g) \equiv \|f - g\|$ ,  $(F, d)$  is a Banach space. We denote the operators  $T, Cc$  that take any function  $f$  in  $F$ , and return functions which for any  $(V, z, q) \in \mathbb{R}_U \times Z \times Q$ , have the values

$$Cc(f)(V, z, q) = \min_{y \in \Gamma(V, z, q)} \left\{ -S(V, z, q) + q \left[ V - \beta(1 - \delta) \sum_{z', q' \in Z \times Q} y(z', q') P(z', q' | z, q) \right] \right\} \quad (46)$$

$$+ q(1 - \delta) \sum_{z', q' \in Z \times Q} f(y(z', q'), z', q) P(z', q' | z, q) \quad (47)$$

and

$$T(f)(V, z, q) = \min_{(\lambda, V^0, V^1) \in \Gamma_E(V, z, q)} \left\{ \lambda(V^0 - U - L) + (1 - \lambda) Cc(f)(V^1, z, q) \right\} \quad (48)$$

$l(x, z, q) \equiv -L - U + x$  is the cost of the bank for liquidating the firm.

We make use of convex and submodular functions. Comprehensive treatment of these topics are in Rockafellar [48] and Topkis [51]. We summarize the results used here:

Let the correspondence (multivalued mapping)  $\partial f$  denote the *subdifferential* of  $f$ , where  $\partial f(x)$  denotes the set of subgradients of  $C$  at  $x$ .  $f$  is differentiable at  $x$  iff  $\partial f(x)$  is a singleton. A convex function  $f$  is almost everywhere differentiable; the left and right derivatives,  $f'_-, f'_+$  satisfy  $f'_-(x) \leq f'_+(x)$  all  $x \in X$ .  $f'_-$  is left-continuous and  $f'_+$  is right continuous. For  $x_1 < x < x_2$ , then  $f'_+(x_1) \leq \partial f(x) \leq f'_-(x_2)$ ; if  $f$  is strictly convex, the two inequalities are strict

Let  $\leq$  be a partial ordering defined on a set  $X$ . For  $x_1, x_2 \in X$  the operations  $\vee$  and  $\wedge$  are defined as  $x_1 \vee x_2 = \inf\{x | x_1 \leq x \text{ and } x_2 \leq x\}$  and  $x_1 \wedge x_2 = \sup\{x | x \leq x_1 \text{ and } x \leq x_2\}$ . A set  $X$  with the partial ordering  $\leq$  is a lattice if for all  $x_1, x_2 \in X$  then both  $x_1 \vee x_2$  and  $x_1 \wedge x_2$  are in  $X$ . Given two sets  $A, B \subset X$ ,  $B$  is said to be *higher* than  $A$ , denoted  $A \sqsubseteq B$  if for  $x_1 \in A, x_2 \in B, x_1 \vee x_2 \in B$  and  $x_1 \wedge x_2 \in A$ . We say that  $B$  is strictly higher,  $A \sqsubset B$  if the previous condition holds and  $A$  and  $B$  are disjoint. A function  $f : X \rightarrow R$  is submodular if for all  $x_1, x_2 \in X, f(x_1 \vee x_2) + f(x_1 \wedge x_2) \leq f(x_1) + f(x_2)$ .  $f$  is said to be strictly submodular if for all unordered  $x_1, x_2 \in X$ , i.e.  $x_1 < x_1 \vee x_2$  and  $x_2 < x_1 \vee x_2$  and  $x_1 < x_1 \wedge x_2$  and  $x_2 < x_1 \wedge x_2$ , then  $f(x_1 \vee x_2) + f(x_1 \wedge x_2) < f(x_1) + f(x_2)$ . A function  $f$  is (strictly) supermodular in  $X$  if  $-f$  is (strictly) submodular in  $X$ .

## A.2 Properties of the Value Function

### Proof of Proposition 2, First Part

The cost of liquidation,  $-L + V - U$  is an unbounded function and for any  $f \in F Cc(\cdot)$  is also unbounded. We use a restricted version of the recursive problem, and later verify that the restriction is not binding.

Consider the Bellman Equation in the text, but impose an arbitrary upper bound,  $B < \infty$  on the admissible set of promise utilities. For any  $(V, z, q) \in [U, B] \times Z \times Q$ , the value function  $C^B$  in this restricted problem must satisfy the same Bellman Equation, but with the additional restriction that  $G_{z',q}(V, z, q) \leq B$ . Define  $T^B$  the functional operator associated with the Bellman Equation is monotone and that because  $q(1 - \delta) < 1$ , discounting holds. Thus  $T^B$  is a contraction on  $(F^B, d)$ , where  $F^B$  restricts the domain of the functions to  $[U, B]$ .

Below, we use the assumption of  $q < \beta$  to find a  $B < \infty$  such that for any  $V \in R_U$  the constraint does not bind. Using the unique fixed point of  $T^B$ , we can also write the formula for the unique fixed point  $C$  of the operator  $T$ , for any  $V \in R_U$ .

Convexity holds by definition of optimal lotteries. Since  $S$  is strictly increasing in  $z$  so is the fixed point. The details are standard and omitted.

### Proposition 3: $\partial C \leq 1$ and $\partial Cc(V, z, q) \leq q$

We first show the second part. Pick  $z, q$  and let  $V_0 < V_1$  and  $f \in F$ . Given  $f$  let also  $y^0$  be the optimal policies given  $(V, z, q)$ .  $y^0 \in \Gamma(V_0, z, q)$ , and therefore, if  $Cc^0(f)(V_1, z, q)$  denotes the continuation value at state  $(V, z, q)$  but restricted to use the policy  $y^0$ , then  $Cc^0(f)(V_1, z, q) \geq Cc(f)(V_1, z, q)$ . Therefore

$$Cc(f)(V_1, z, q) - Cc(f)(V, z, q) \leq Cc^0(f)(V_1, z, q) - Cc(f)(V_0, z, q) \quad (49)$$

$$= -S(V_1, z, q) + qV_1 - [-S(V_0, z, q) + qV_0] \quad (50)$$

$$\leq q(V_1 - V_0) \quad (51)$$

$$(52)$$

as desired. The first part is immediate.  $\square$

### Proposition 4

We already know that the fixed point is convex. Consider a function  $f$  convex and submodular in  $(V, z)$ . For simplicity assume that  $\partial f$  is a singleton  $f_1$  everywhere. Fix  $V, q_0$ , two states  $z_0 < z_1$  and, consider two other  $z^0 < z^1$  in which there is not liquidation, given next interest rate  $q_1$ . Then,

$$\begin{aligned}
\partial Cc(f)(V, z_0, q) &= -\frac{\partial S(V, z_0, q_0)}{\partial V} + q_0 + q_0(1 - \delta)f_1(G_{z^0, q_1}(V, z_0, q_0), z^0, q_1) \\
&\geq -\frac{\partial S(V, z_1, q_0)}{\partial V} + q_0 + q_0(1 - \delta)f_1(G_{z^0, q_1}(V, z_0, q_0), z^0, q_1) \\
&\geq -\frac{\partial S(V, z_1, q_0)}{\partial V} + q_0 + q_0(1 - \delta)f_1(G_{z^0, q_1}(V, z_1, q_0), z^0, q_1) \\
&= \partial Cc(f)(V, z_1, q)
\end{aligned} \tag{53}$$

The first equality derives by using the envelope condition and the assumption that  $G_{z^0, q_1}(V, z_0, q_0) > U$ ; the second from the fact that  $z_1 > z_0$  and that  $-S$  is submodular. The second inequality requires more argument. First, our assumption that  $G_{z^0, q_1}(V, z_0, q_0) > U$  and  $G_{z^1, q_1}(V, z_0, q_0) > U$ , then optimality requires  $f_1(G_{z^0, q_1}(V, z_0, q_0), z^0, q_1) = f_1(G_{z^1, q_1}(V, z_0, q_0), z^0, q_1)$ ; but since  $f$  is submodular, then,  $(G_{z^0, q_1}(V, z_0, q_0), z^0, q_1) \leq (G_{z^1, q_1}(V, z_0, q_0), z^0, q_1)$ . Therefore, given submodular  $f$ , the optimal continuation policies are increasing in  $z'$ ; But since  $P_z(z', z)$  is monotone in  $z$ , the non-negativity constraint of consumption requires that  $(G_{z', q_1}(V, z_1, q_0), z^0, q_1) \leq (G_{z', q_1}(V, z_0, q_0), z^0, q_1)$  all  $z'$ . Using this, the second inequality follows from the convexity of  $f$ . The last equality holds again from using the envelope condition, which completes the argument for  $Cc(f)$  to be submodular.  $\square$

### A.3 Continuation Policies

From the convexity of  $C$  it follows that the first order conditions are necessary and sufficient for the continuation policies. They are given by

$$\beta(1 - \mu_2(V, z, q)) + \mu_1(z', q'; V, z, q) \in \partial C(G_{z', q'}(V, z, q), z', q'), \quad \forall (z', q') \in Z \times Q \tag{54}$$

$$\mu_1(z', q'; V, z, q) \geq 0; \quad G_{z', q'}(V, z, q) \geq U; \text{ and at least one with equality } \forall (z', q') \in Z \times Q \tag{55}$$

$$\mu_2(V, z, q) \geq 0, \quad \beta(1 - \delta) \sum_{z', q'} G_{z', q'}(V, z, q) P(z', q' | z, q) \leq V, \text{ and at least one with equality} \tag{56}$$

where  $\mu_1(z', q'; V, z, q)$  and  $\mu_2(V, z, q)$  are  $\#Z \times \#Q + 1$  (scaled) Kuhn-Tucker multipliers.

#### Proof of Proposition 5: $\mathbf{G}$ is non-decreasing in $\mathbf{V}$

Fix  $(z, q)$  and let  $V_0 < V_1$ . For all those  $z', q'$  s.t  $\mu_1(z', q', V, z, q) > 0$  the proposition holds trivially as  $G_{z', q'}(V_0, z, q) = G_{z', q'}(V_1, z, q) = U$ . Also, if  $\mu_2(V_0, z, q) = \mu_2(V_1, z, q) = 0$  then  $G_{z', q'}(V_0, z, q) = G_{z', q'}(V_1, z, q)$ , as the choice is not restricted by  $V_0, V_1$ . Assume now that  $\mu_2(V_0, z, q) > 0$  and hence  $\beta(1 - \delta) \sum_{z', q'} G_{z', q'}(V, z, q) P(z', q' | z, q) = V_0$ . Assume to the contrary of the proposition that  $\exists z^*, q^*$  s.t.

that all the optimal choice for  $z^*, q^*$  is strictly at state  $(V_0, z, q)$  than given  $V_1, z, q$ , i.e.  $G_{z^*, q^*}(V_1, z, q) \sqsubset G_{z^*, q^*}(V_0, z, q)$ . If that is the case, necessarily  $C(\cdot, z^*, q^*)$  cannot have constant slope in the entire region  $[\max\{G_{z^*, q^*}(V_1, z, q)\}, \min\{G_{z^*, q^*}(V_0, z, q)\}]$ . This implies that  $\partial C(G_{z^*, q^*}(V_1, z, q), z^*, q^*) \sqsubset \partial C(G_{z^*, q^*}(V_0, z, q), z^*, q^*)$

Using the FOC for  $(V_0, z, q)$  and  $(V_1, z, q)$  is direct to conclude that  $\partial C(G_{z', q'}(V_1, z, q), z', q') \sqsubset \partial C(G_{z', q'}(V_0, z, q), z', q')$  all  $z', q'$ .

Since  $\mu_2(V, z, q) > 0$  it cannot be the case that there is a policy selection  $\hat{G}$  such that  $\hat{G}_{z', q'}(V_0, z, q) \leq G_{z', q'}(V_0, z, q)$  for all  $z', q'$  and that for some the inequality is strict (otherwise the FOC could be satisfied with  $\mu_2(V_0, z, q) = 0$ ). Therefore, it must be the case that  $G_{z', q'}(V_0, z, q) \geq G_{z', q'}(V_1, z, q)$  all  $z', q'$  and for some  $z', q'$  the inequality being strict. But that contradicts the optimality of  $G_{z', q'}(V_1, z, q)$  as  $G_{z', q'}(V_0, z, q) \in \Gamma(V_1, z, q)$   $\square$

### Proof of Proposition 6

Let  $(V, z, q)$  be the initial state, fix any  $q' \in Q$  and let  $z_0 < z_1$ . If both  $\mu_1(z_1, q'|V, z, q) > 0, \mu_1(z_0, q'|V, z, q) > 0$  the conclusion trivially holds. Consider the case  $\mu_1(z_1, q'|V, z, q) = \mu_1(z_0, q'|V, z, q) = 0$ . The FOC imply that

$\partial C(G_{z_0, q'}((V, z, q), z_0, q')) \sqsubseteq \partial C(G_{z_1, q'}((V, z, q), z_1, q'))$ . The proposition holds from the convexity and submodularity of  $C$ . In the regions where  $G_{z, q}(\cdot) > U$ , the probability of continuation is positive, and  $Cc$  is strictly decreasing in strictly submodular in  $V, z$ , and therefore the order is strict.

### Proof of Proposition 8

The operator in the RHS defines a contraction on the space of bounded functions; hence  $D < \infty$  and is unique. Since  $M$  is strictly increasing in both arguments and  $P(\cdot, \cdot|z, q)$  is monotone, then the fixed point  $D$  is also strictly increasing in both arguments too. That  $D(\bar{z}, \bar{q}) = M^*(\bar{z}, \bar{q})$  follows immediately from  $D(\bar{z}, \bar{q}) > \beta(1 - \delta)D(z', q')$  all  $z', q'$ .

### Proof of Proposition 9

For the second part, because  $q \geq \beta$ , the FOC indicate that either  $G_{z', q'}(V, z, q) = M(z', q')$  or that  $\mu_2(V, z, q) > 0$ . by convexity policies outside these regions cannot intersect with the region where the probability of liquidation is positive, because in the latter region  $C$  has a constant slope lower than  $q$ . The argument for the strict submodularity of  $Cc$  on  $(V, z)$  follows the same lines.

### Proof of Proposition 10

Assuming that  $\lambda \in (0, 1)$ . Since  $Cc$  has a slope lower or equal to  $q < 1$ , it is optimal to make  $V^1$  as large as possible. Then, if  $\lambda \in (0, 1)$ , then  $V^0 = U$  and  $\lambda = \frac{V^1 - U}{V^1 - U}$ . Substituting this expression, and optimizing with respect to  $V^1$  gives the expression in the proposition. The LHS is strictly decreasing as  $\partial Cc \leq q$ , and the RHS is increasing. Then if there is a finite  $V^1$  satisfying the equation, it is unique. The conclusion that such  $V^1$  must be necessarily less or equal to  $M(\bar{z}, \bar{q})$  follows from the fact that  $C$

has constant slope  $q < 1$  for  $V > M^*(\bar{z}, \bar{q})$ . Thus, if no value less or equal to  $M(\bar{z}, \bar{q})$  can satisfy that expression, then a degenerate solution  $V^1 \rightarrow \infty$  and  $\lambda \rightarrow 1$  is optimal.

## End of Proof of Proposition 2

When the upper-bound  $B \geq M(\bar{z}, \bar{q})$ , then, the optimal continuation policies in the restricted problem coincide with those in the general problem. Then, the unique fixed solution of  $C$  is as follows: for any  $(V, z, q)$ , if  $V \in [U, B]$ ,  $C(V, z, q) = C^B(V, z, q)$ . For  $V > B$ ,

$$C(V, z, q) = \min\left\{-L + V - U, Cc^B(V, z, q) + q(V - B)\right\}$$

## Proof of Proposition 12

If  $\mu_1 > 0$  then  $G_{z', q'}(V, z, q) = U$ , and the result holds trivially. Assume then  $\mu_1 = 0$ . First consider the case with assume  $\mu_2 = 0$  (the limited liability does not bind). In the first order conditions, the first term is the discount factor  $\beta$  while the second is the would be a constant in the region of randomizations, and strictly increasing outside of it until achieving  $q$ . If the discount factor is strictly higher, then the optimal plan requires  $G_{z', q'} = U$  while if it is strictly lower, then  $G_{z', q'}(V, z, q) = V^1(z', q')$ . In, zero probability even that  $\beta \in \partial C(V^1(z', q'), z', q')$ , then select either extreme, say  $U$ . (and therefore the agent consumes in the present date). If limited liability holds of equality, either extreme can be chosen, as the entrepreneur and the bank are indifferent in the timing of the transfers. If the limited liability binds, repeat the argument with  $\beta(1 - \mu_2)$ .

## Proof of Proposition 13

As long as  $K > L$ , it is easy to prove that it is never optimal to create a firm that will be liquidated with positive probability in the very first period. Then, necessarily newly created firms satisfy  $Cc(V_0, z_0, q_0) + K_0 = 0$ . The first part of the proposition follows as  $Cc$  is globally strictly decreasing with respect to  $z$ , and in the relevant region, strictly increasing in  $V$ . Now for a given  $L$ , for a large enough  $K_0 > L$ , there exist  $0 < \delta_{K_0}$  and  $0 < \beta_{K_0} < 1$  s.t. for all  $0 < \delta < \delta_{K_0}$  and  $\beta < [\beta_{K_0}, 1)$ , it is the case that whenever  $Cc(V_0, z, q) = -K_0$  then

$$q \left[ V_0 + (1 - \delta) \sum_{z', q'} [C(G_{z', q'}(V_0, z, q), z', q') - \beta G_{z', q'}(V_0, z, q)] P(z', q' | z, q) \right] < 0$$

This conditions simply says that the loan cannot be fully recovered from the repayment in the first period of the firm. In this case, higher  $q$  unambiguously reduces the value  $Cc(V_0, z, q)$  for all  $V_0$  and since  $Cc(V_0, z, q)$  is strictly increasing –in the region of initialization– then the breaking even condition requires  $V_0(z, q)$  to increase.

## Proof of Proposition 14

Fix  $(V, z, q)$  and assume that  $V \geq V_0(z, q)$ . Notice that independently of the value of  $\mu_2(V, z, q)$ , for those realizations  $(z', q')$  such that  $\mu_1(z', q'|V, z, q) > 0$  the optimal policy is  $G_{z', q'}(V, z, q) = U$  and the proposition follows directly.

From now on, assume that  $\mu_1(z', q'|V, z, q) = 0$ . Consider first the case in which  $\mu_2(V, z, q) = 0$ . Then the first order condition implies that

$$\beta \in \partial C(G_{z', q'}(V, z, q), z', q') \quad (57)$$

but since  $C$  is convex, then for any  $\epsilon > 0$

$$\partial C(G_{z', q'}(V, z, q) + \epsilon, z', q') \geq \beta > 0 \quad (58)$$

and therefore, a positive increment in the utility entitlement of the entrepreneur induces a strictly positive increment in the cost for the entrepreneur, as claimed in the proposition. Finally, consider the case  $\mu_2(V, z, q) > 0$ . The envelope condition is

$$\partial Cc(V, z, q) = -\frac{\partial S(V, z, q)}{\partial V} + q(1 - \mu_2(V, z, q)) \quad (59)$$

Since  $Cc$  is convex and  $V \geq V_0(z, q)$  then  $\partial Cc(V, z, q) \geq \partial Cc(V_0(z, q), z, q) > 0$ . Thus, necessarily  $\mu_2(V, z, q) < 1$ . But then, from the FOC and for any  $\epsilon > 0$

$$\partial C(G_{z', q'}(V, z, q) + \epsilon, z', q') \geq \beta(1 - \mu_2(V, z, q)) > 0 \quad (60)$$

and the argument is complete.

## Allocations in $\mathcal{P}(\mathcal{A}, \mathcal{B}, p^*)$ replicate $\sigma$

**Deterministic case** Assume that there is an allocation  $\{\bar{c}_t, \bar{a}_t, \bar{k}_t\}$  that dominates  $\sigma$  and satisfies  $\mathcal{P}(\mathcal{A}, \mathcal{B})$ . Consider first the case where  $q = \beta$ . There is a  $t_0$  such that  $\sum_{t \geq t_0} \beta^t c_t^\sigma < \sum_{t \geq t_0} \beta^t \bar{c}_t$ , with  $\bar{c}_{t_0} > c_{t_0}^\sigma$ . Let  $\epsilon_{t_0} \equiv \bar{c}_{t_0} - c_{t_0}^\sigma$ . By construction  $\sum_{t \geq t_0} \epsilon_t > 0$ . Therefore, it is the case that the optimal long term contract at time  $t_0$  could have achieved the utility  $\sum_{t \geq t_0} \beta^t c_t^\sigma$  with the resources  $A_t - \sum_{t \geq t_0} \epsilon_t < A_t = C_t^\sigma$  which is absurd given the definition of  $C_t^\sigma$ . But since in the case of  $\beta = q$  there are infinitely many solutions to the recursive optimal financial relationships, then there can also be infinitely many optimal solutions to  $\mathcal{P}(\mathcal{A}, \mathcal{B})$ . Consider now the case  $q > \beta$ ; now the allocation solving the recursive long term contract is unique. By of discounting, the far future can be neglected, so, without loss of generality, we can assume that there is a finite  $n$ , such that  $\bar{s} = \bar{c}_t, \bar{a}_t, \bar{k}_t \ t \geq n = \{c_t^\sigma, a_t^\sigma, k_t^\sigma : t \geq n\}$ . That  $\bar{s}$  dominates  $\sigma$  means that  $\sum_{0 \leq t \leq n} \beta^t c_t^\sigma < \sum_{0 \leq t \leq n} \beta^t \bar{c}_t$ . If  $\bar{c}_t \geq c_t^\sigma$  all  $t$  and at least in one period the inequality is strict, we obtain a contradiction with the construction of  $\mathcal{A}$ . The only possibility is that there are at least two dates  $\tau_1 < \tau_2$  such that the in one date the consumption is lower while in the other it is higher. Because  $\sigma$  is the fastest repayment, then,  $c_{\tau_1}^\sigma > c_{\tau_2}^\sigma$ . Then,  $\bar{a}_{\tau_1+1} = A_{\tau_1+1} - q[c_{\tau_1}^\sigma - c_{\tau_2}^\sigma] < A_{\tau_1+1}$ , so  $\bar{s}$  is not feasible in  $\mathcal{P}(\mathcal{A}, \mathcal{B})$ .

## B A General Equilibrium Interpretation

In this section I provide an interpretation of the model as the general equilibrium of a closed economy with four types of agents: households, potential entrepreneurs, banks and a government.

Asset markets play a crucial role in this economy because they determine the opportunity cost of the resources used by active entrepreneurs, influencing as well, the optimal decisions for creation and liquidation of firms. One can view this economy as one in which all, banks, government and households have access to the access to frictionless asset markets. Assuming that households own all the shares of the banks, I can restrict attention to the trading between the government and households.<sup>27</sup> I shall consider economies whether the government only trades in one-period riskless bonds, and thus, with symmetric households, bonds will be the only asset that needs explicit pricing. While each household takes  $\{q\}$  this process as given, the government is a large agent whose portfolio decisions can affect it. We assume that random portfolio/expenditure decisions of the government are the only source of uncertainty in the economy.

### Households

There is a continuum (with unitary total mass) of identical, infinitely lived households. Households own all the shares of the banks and with no loss of generality I assume that each period they receive (or pay for) the surplus  $s_t$  of the banking intermediation sector. They have another primary source of income, a positive constant flow  $y_H$  of the consumption/investment good.

$$E \left\{ \sum_{t \geq t_0} \beta_h^t \left[ \frac{(c_t^h)^{1-\rho}}{1-\rho} \right] \mid H_{t_0} \right\} \quad (61)$$

where  $\beta_h \in (0, 1)$ ;  $\rho > 0$ ; and  $H_{t_0}$  is the household's relevant information as of  $t_0$ . Every period, each household produce  $l_t$ , receives the banks net surplus  $s_t$  and pays lump-sum taxes  $\tau_t$  to the government. Letting  $a_t$  be the units of consumption good bought at time  $t - 1$  to be delivered at time  $t$ , the budget constraint of the households are given by the sequential constraint,

$$q_t a_{t+1} + c_t^h = y_H + s_t - \tau_t + a_t \quad (62)$$

and the transversality constraint

$$\forall t, \lim_{T \rightarrow \infty} \left[ \prod_{j=0}^{T-1} q_{t+j} \right] a_{t+T} = 0 \quad a.s. \quad (63)$$

Households take  $q_t, s_t, \tau_t$  as given and maximize their utility by choosing  $c_t, l_t$  subject to the sequential and transversal constraints.

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<sup>27</sup>An alternative is to allow banks to trade bonds while also allowing households to trade shares of the banks. This alternative adds nothing but notation.

## Government

The government consumes non-negative amounts  $g_t$  of the good. The sequential budget constraint of the government is

$$q_t b_{t+1} + g_t = \tau_t + b_t \text{ all } t$$

and the transversality constraints

$$\lim_{T \rightarrow \infty} \left[ \prod_{j=t}^{T-1} q_j \right] b_T = 0, \text{ a.s.} \quad (64)$$

The maintained assumption in this paper is that the policy of the government is **exogenous** in the sense that **two** out of  $\{b_{t+1}, \tau_t, g_t, q_t\}$  are exogenous stochastic process. More specifically, these two processes may dependent on each other, but are independent of any other variables in the economy. The other two components of the government must satisfy the market clearing requirement (budget constraints and optimality of households.)

While the original source of aggregate uncertainty in this economy is from the processes  $\{b_t, \tau_t\}$ , we follow a "back-solving" strategy of assuming that these processes are set such that  $\{q_t\}$  follows an exogenous, stationary, ergodic Markov process.

The government follows a policy of fixing the interest rates, as appears to be the case in many actual economies. Here, the government adjust the "quantities" in its control so as to sustain  $\{q_t\}$  as an equilibrium price process.<sup>28</sup> The assumptions on the borrowing and lending of the government is simply to guarantee that effectively the government can sustain  $\{q_t\}$  as an equilibrium process.

I now formulate the consolidated surplus (deficit) for the banks as a whole. First, bankers receive the repayment from the entrepreneurs active the previous period. They can also receive net resources (not necessarily positive) from liquidating firms. In addition, they need positive resources for the newly created firms as well as for the working capital for all active firms. The relevant expressions are:

## Repayments

$$R_t = R(q_{t-1}, \psi_{t-1}) = \int_Z \int_{\mathbb{R}_U} \{z f(k(V, z, q_{t-1})) + \beta(1 - \delta)g(V, z, q_{t-1}) - V\} \psi_{t-1}(dV, dz) \quad (65)$$

## Resources from Liquidations

$$L_t = L(q_{t-1}, \psi_{t-1}) = \int_Z \int_{\mathbb{R}_U} l(V, z, q_t) \{U + L - V\} [T_E(q_{t-1}) \psi_{t-1}](dV, dz) \quad (66)$$

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<sup>28</sup>As opposed to the traditional equilibrium models that impose that given an exogenous process for the quantities of the government, the prices must adjust to clear markets. Here is the converse.

## Set-Up Costs of New Firms

$$I(q_t) = \delta K_0 m [T_c(q_t)] \quad (67)$$

## Working Capital

$$K_t = K(q_{t-1}, \psi_{t-1}) = \int_Z \int_{\mathbb{R}_+} k(V, z, q) \psi_t(dV, dz) \quad (68)$$

where the last equation uses the fact that  $\psi_t$  is determined by  $(\psi_t, q_t)$ .

Thus, the net surplus for the consolidated banking sector is

$$s(\psi_{t-1}, q_t, q_{t-1}) = R(\psi_{t-1}, q_{t-1}) + L(\psi_{t-1}, q_t, q_{t-1}) - I(q_t) - K(\psi_{t-1}, q_{t-1}) \quad (69)$$

## Equilibrium

Recall that households receive every period the net surplus  $s_t$  from the banks. At this point it must be clear that the information set  $H_t$  contains all the relevant information regarding  $\tau_t, q_t, \psi_t$  including possibly  $b_t, g_t$ . Given  $H_t$ , and taking as given  $q_t, \tau_t, s_t$ , the sufficient conditions for the optimal choice of  $l_t, c_t, a_{t+1}$  are the first order conditions

$$(y_H + s_t - \tau_t + a_t - q_t a_{t+1})^{-\rho} = \frac{\beta_h}{q_t} E[(y_H + s_{t+1} - \tau_{t+1} + a_{t+1} - q_{t+1} a_{t+2})^{-\rho} | H_t]$$

and the transversality constraint discussed before.

**Definition 4. Equilibrium** *Given the primitives on households  $(\beta_h, \rho, \kappa_0, v)$ , entrepreneurs and their technologies  $(\beta, \delta, f, P_z, L, U, K_0)$  and government policy  $P_q$ , and the initial conditions  $a_0, \psi_{-1}, q_{-1}$ , an equilibrium in this economy consists of (a) a household allocation  $\{a_{t+1}, c_t : t \geq 0\}$ , (b) a government policy process  $\{\tau_t, g_t, b_{t+1}, q_t : t \geq 0\}$ , (c) Policy and value functions of individual contracts  $(g, C)$  and  $V_0$ , (d) aggregate operators  $T_L, T_E, T_C$  of creation, endurance and destruction, (e) a sequence of measures  $\{\psi_t : t \geq 0\}$  and a sequence of net surpluses  $\{s_t : t \geq 0\}$ , such that:*

1. **Optimal Contractual Relationships** *The functions  $(g, C)$  solve the optimal recursive contract for ongoing bank-entrepreneur relationships.*
2. **Competition in Intermediation Market** *Given the functions  $(g, C)$ , and for every pair  $(z, q)$ , the firms are activated if and only if the bank obtains non-negative expected payoffs and in that case, the initial utility of the entrepreneur is  $V_0(z, q) \equiv \arg \sup_{V \geq U} \{C(V, z, q) + K_0 \leq 0\}$ .*
3. **Aggregation Consistency** *The aggregate operators  $T_L, T_E, T_C$  are derived from the functions  $(g, V_0)$ , and the transitions  $(P_z, P_q)$*

4. **Consistent Aggregate Dynamics** The process  $\{q_t\}$  has transitions  $P_q$  and the transition for  $\{\psi_t\}$ , is given by

$$\psi_t = (1 - \delta)[T_E(q_t)\psi_{t-1}] + \delta T_C(q_t)$$

The banking surplus is given by  $s_t = s(\psi_{t-1}, q_{t-1}, q_t)$  as defined above.

5. **Bond Markets Equilibrium** Given  $a_0, \psi_0$  and for any realization of  $\{q_t : t \geq 0\}$ , and the implied  $\{s_t : t \geq 0\}$ , the realization of the processes  $\{\tau_t, b_{t+1} : t \geq 0\}$  is such that:

(a) Taking  $\{s_t, \tau_t : t \geq 0\}$  as given,  $\{c_t^h, a_{t+1} : t \geq 0\}$  solves the problem of households.

(b) The quantity of government bonds clears the markets,  $a_t + b_t = 0$  all  $t$ .

6. **Government Policy Feasibility** Given  $a_0$ , the sequences  $\{q_t, \tau_t, b_{t+1}, g_t : t \geq 0\}$  satisfy the sequential and transversal constraints of the government and  $g_t \geq 0$  all  $t$ .

Imposing additional restriction on  $\{q_t\}$  one can obtain the fiscal policy shocks  $\{g_t, \tau_t, b_{t+1}\}$  so that all the market clearing, individual maximizations and budget constraints are satisfied. Because of the Ricardian Equivalence, many  $\{\tau_t, b_{t+1}\}$  are compatible with  $\{q_t\}$  and what matters is the sequence  $\{g_t\}$ .

## C Computational Algorithm

Here we discuss in some the detail the steps followed to obtain the aggregate time series. First, the optimal policies (continuation, liquidation and activation decisions) are computed. Second, from the liquidation, continuation and activation operators are computed. Finally, the realizations of the interest rates and the initial conditions of the economy are explained.

### C.1 Computing Individual Contracts

As explained above, I assume that both shocks in the economy follow Markov chain process that approximate AR(1) Gaussian processes. As explained in the text, I employ the methods of Tauchen and Haussey[53] to obtain the approximation. Thus, the support for the exogenous states for relationships are  $Z \times Q$ , and the optimal contract solves a dynamic programming problem with a finite configuration  $Z \times Q$  of exogenous states and a continuous endogenous state  $V$ . From the paper, we know that continuous policies are contained in the compact set  $[U, M(\bar{z}, \bar{q})]$

In this problem we have a known linear function for the underlying value of the option of liquidation. Moreover, the liquidation value is independent of the realization of the exogenous shocks. Exploiting the finite number of shocks, I parameterize the *continuation* cost of the relationship. For each  $(z, q, V) \in Z \times Q \times R_U$  the continuation cost is

$$C_{Ca}(V, z, q) \equiv A(z, q)^t T[\varphi(V)] \quad (70)$$

where  $A(z, q)$  is a column vector  $Np \times 1$ , and the superscript  $t$  denotes transposition;

$$T(x) = [T_0(x), T_1(x), T_2(x), \dots, T_{Np-1}(x)]^t$$

Figure 10: Continuation and Liquidation Cost Function

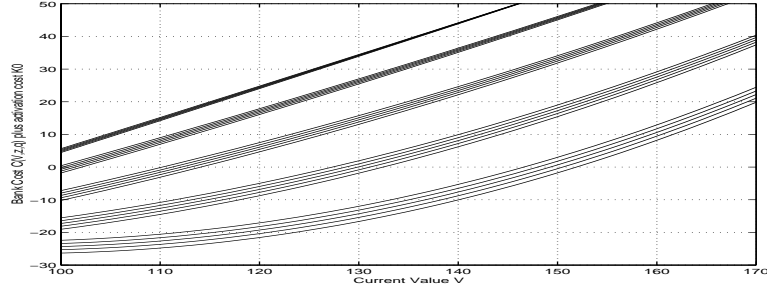
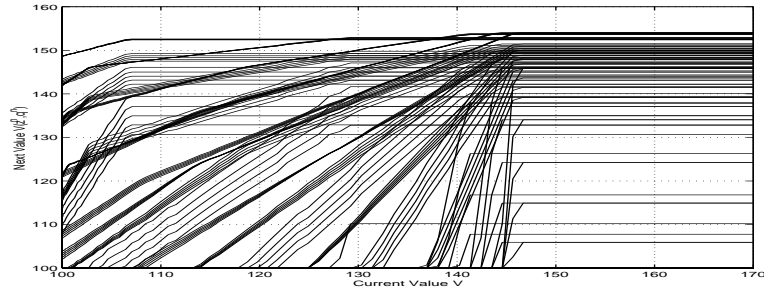


Figure 11: Continuation Policy Profile for a given state  $(z, q)$



where  $T_n(\cdot)$  is the  $n$ -th Chebyshev polynomial, i.e.  $T_0(x) = 1$ ;  $T_1(x) = x$  and for  $n \geq 2$ ,  $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ . Chebyshev polynomials are defined on  $x \in [-1, 1]$ . Then the bijective function  $\varphi(V) = 2\frac{V-U}{M(\bar{z}, \bar{q})-U} - 1$  is used to map  $[U, M(\bar{z}, \bar{q})]$  into  $[-1, 1]$ . Here  $N_p$  is the number of polynomials used. It is well known that the family  $\{T_n : n \geq 0\}$  is a basis for all  $C[-1, 1]$ . The fixed point functions  $C, Cc$  are known to be continuous, and approximations with high values of  $N_p$  are virtually exact.

Given any value  $A$ , it is not hard to derive an array of values  $a$  such that

$$Cc(V, z, q) = \sum_{j=0}^{N_p-1} a_j(z, q)V^j \quad (71)$$

Given the  $a$ , then it is easy to compute the continuation, liquidation and activation decisions.

## Activation

Find the maximum root of the polynomial  $K0 + \sum_{j=0}^{N_p-1} a_j(z, q) * V^j$ . If that root is greater than  $U$  the firm is activated. Otherwise, the entrepreneur goes to the background.

## Liquidation

First, we compute the continuation value  $V^1(z, q)$  for entrepreneurs that win the lottery. Given  $a(z, q)$  such value is given by the highest root of the polynomial

$$K0 + \sum_{j=0}^{Np-1} (1-j)a_j(z, q)V^j$$

If  $V > V^1(z, q)$  the entrepreneur continues with probability one.

## Continuation

For each state  $(V, z, q)$ , the computation of the continuation policies is as follows. First, if for a given  $(z', q')$ ,

$\beta < \sum_{j=0}^{Np-1} ja_j(z, q)(V^1(z, q))^{j-1}$  then the prescribe policy is  $G_{z', q'}(V, z, q)$  for all  $(V, z, q)$ . On the other extreme, when the limited liability constraint is not binding, the continuation choice is a root of the polynomial

$$\beta - \sum_{j=0}^{Np-1} ja_j(z', q')V^j \quad (72)$$

Finally, it might be the case that the limited liability constraint binds at the state  $(V, z, q)$ . In general we proceed as follows: we pick one state such that if the firm continues in any state in the future, it will continue in that state. Therefore, we pick the state  $\{\bar{z}, \bar{q}\}$ . We consider a very fine grid on  $Vg \subset [U, M(\bar{z}, \bar{q})]$  of possible values assigned to that state. For each of the grid points  $Vg(n)$ , then the choice for the other states  $\{z', q'\}$  is given by a root of

$$\sum_{j=0}^{Np-1} ja_j(z', q')V^j - \sum_{j=0}^{Np-1} ja_j(\bar{z}, \bar{q})Vg(m) \quad (73)$$

Then, for each state  $(V, z, q)$ , we select the set of points of  $Vg$  such that the implied profile of continuation values satisfy the limited liability constraint. The optimization is thus one dimensional, and given the parameters  $a$  it consists on selecting the element in the grid that minimizes the continuation cost.

## Iterating the Algorithm

Thus, given an array of values  $A$  one can derive the continuation, liquidation and activation decisions. The problem is then to obtain the values  $A(z, q)$  that yield a good approximation to  $Cc$ . The steps required and how I proceeded are as follows:

Select a grid of  $M \geq Np$  points on which evaluate the value function. The grid is taken to be the zeroes of the  $M$ -th degree polynomial. (i.e.  $\varphi^{-1}(zeroes)$ ).

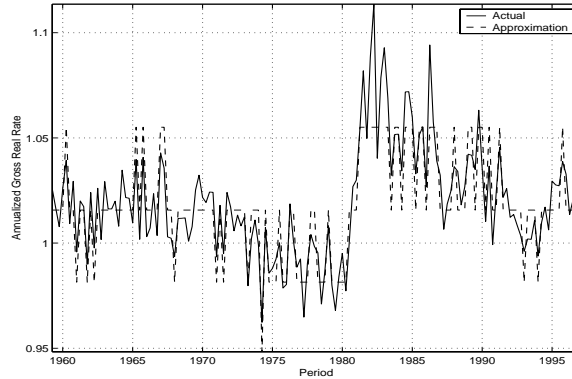
The steps of the iterative algorithm are:

1. For each of the  $M$  grid points and given a value for the vectors  $A_n(z, q)$ , obtain the optimal liquidation and continuation decisions. Obtain the value achieved for each grid point  $Cc(V(m), z, q)$ .
2. Update the parameters  $A_n(z, q)$  by minimizing the square residuals of  $A(z, q)^t T(\varphi(V)) - Cc(V, z, q)$  at the grid points. This step is simple, and yields

$$A_{n+1}(z, q) = (X'X)^{-1}X'Cc_n(:, z, q)$$

where  $X(i, j)$  is the value of the  $j$ -th polynomial,  $j = 1 : Np$  evaluated at the  $i$ -th zero of the  $M$ -th Chebyshev necessarily. To insure numerical stability, use a relaxation parameter for updating  $A_n$ .

Figure 12: Effective and Approximate Real Interest Rate



3. Iterate until convergence.

Notice that the continuation policies **not** restricted to belong to the grid of  $M$ -points. Also, the grid for the candidates values for the continuation values for state  $\bar{z}, \bar{q}$  is chosen to be much finer than the grid for which we are computing the value of  $Cc(\cdot, Z, Q)$ .

A restricted version of this algorithm is to impose that  $Np = 3$ . In this case the functions are quadratic. This brings along a great simplification, as the roots of the first order conditions are unique, and easy to compute, without the need of numerical interpolation. Such restriction, which is used in the experiments reported in the paper, greatly speed up the iterations but work well only when the value function is globally increasing.

## C.2 Computing the Aggregate Operators

The policy function, derived from this dynamic program is the basis for the numerical simulations for the aggregate economy. While the continuation policies were not restricted, in order to keep track of the distribution of firms, in this paper I used a grid of values  $z, V$ . The grids are the states of the Markov chain of  $z$  and the grid of  $M$  points on  $[U, \bar{V}]$ . The experiments use values of  $M$  that range between 70 and 200. Given the equipment available, larger values are computationally unfeasible.

## C.3 Realization of Aggregate Shocks

While the properties of the process  $\{q\}$  are needed to compute the dynamic contract, its realizations are needed to compute the aggregate time series. Naturally, the simulations use the approximation values from the 5-state Markov chain, as the contracts are not define for other realizations.

The figure here displays the effective real interest rate and the implied approximation. The simulations presented in the body of the paper are obtained by running first 500 periods the economy assuming a constant interest rate  $\{q_t = q_3 : 1 \leq t \leq 500$  and then running the economy for the approximated interest rates from 1959:I-1997:II. This arbitrary strategy to select the initial distribution is not giving necessarily the best performance of the economy, In the future, we will simulate the economy for random initial conditions.

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